Nonlinear Elasticity

Course content

- Kinematics of continuum deformation and motion
- Balance laws and equations of motion
- Stress
- Constitutive equations for elastic materials
- Stress-strain equations

Some background references on
Continuum Mechanics and Nonlinear Elasticity

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G.A. Holzapfel,
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R.W. Ogden,
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Y.B. Fu and R.W. Ogden (Eds.),
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University Press, Cambridge 2001.

1 Introduction

1.1 Context

In its beginnings, nonlinear elasticity was seen as an extension of linear elasticity to include “quadratic” terms. With the pioneering and unifying vision of Ronald Rivlin [1915-2005] it came of its own as a discipline in the 1950s to include finite deformations, with no restrictions on their magnitude. Rivlin sought exact solutions to the general equations of motion of an elastic continuum, instead of performing successive approximations. He was primarly motivated by the desire to model the mechanical behavior of rubber when subjected to large extensions. Since the 1980s or so, an intense effort is taking place worldwide to apply, transpose and extend the theories of nonlinear elasticity to the modeling of biological soft tissues such as skin, arterial wall, cardiac muscle, brain, tumors, tendons, etc. Its ultimate goal is to provide accurate simulations for applications in mechanical engineering, bio-engineering, and bio-medicine.

This course aims at showing how the equations of continuum mechanics can be used to provide a satisfactory framework for the description, explanation, and prediction of the mechanical behavior of rubber-like materials and soft tissues, by focussing on their *elastic* response.

Elasticity is a concept that is well understood intuitively. Take an undeformed, unloaded and unstressed elastic body. Apply external forces: it deforms instantly. Remove the forces: it returns to its original size and shape instantly.

From the points of view of experimental physics, chemistry, engineering and biology, it must be kept in mind that elasticity is an *idealization*. Nonetheless, nonlinear elasticity has been, and will prove to be, very useful for many real-world applications including the design and modeling of aircraft structures, bridges, tires, mountings, bearings, earthquake protection devices, etc., and of several bio-materials and real tissues used in bio-engineering and bio-medicine. It also provides a most useful tool in the evaluation, prospection and visualization of the Earth crust and of mechanical and biological structures.

Linear elasticity is concerned with the mechanics of solids for which the stress T is proportional to the strain E in a tensile test, say. Figure 1(a) shows a typical response curve for a metal. There, the point P represents the limit of proportionality, and Y is the yield point (beyond Y the body deforms plastically, that is, it does not return to its original shape and state once the load is removed.) Experimentally, measurements on metals show that P corresponds to a strain in the region 0.01%–0.1%. Consequently, the theory may be linearized and retain only linear terms in the expansions of the equations with respect to the strain E. This is the realm of the so-called infinitesimal, or *classical*, theory of elasticity.

Figure 1(b) shows a typical tensile response for a biological soft tissue. Here, the point Y corresponds to stretches in the ranges 10%–100%. The P–Y part of the curve corresponds to the “strain-stiffening” effect. It is often modeled with a

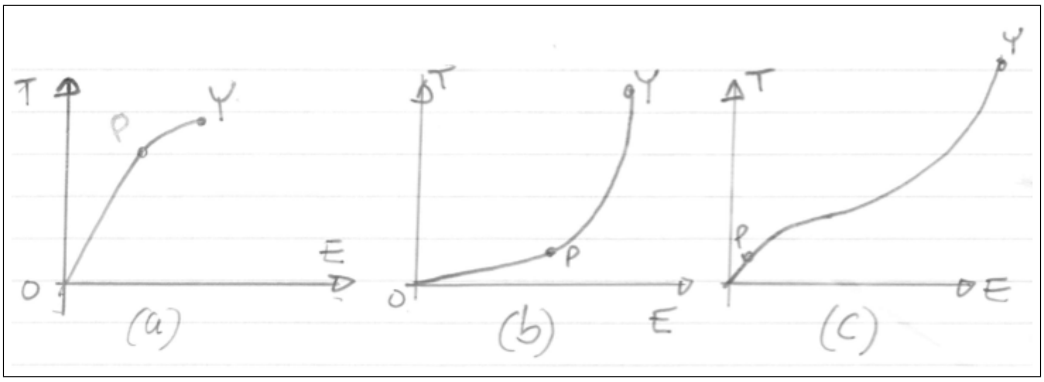

Figure 1: Typical stress-strain response curves in tensile tests of (a) a metal, (b) a soft tissue, (c) a rubber.

exponential type of function. The regime O–P can be much wider than for metals, and can go up to 5%. Here linearization in terms of E is not appropriate. The strains are *finite*.

Figure 1(c) shows a typical tensile response for rubber-like materials. Rubbers and elastomers can stretch up to 100%–1500%. Clearly here, linear elasticity can only cover the very early part of the curve.

The initial slope of the tensile tests plots gives a measure of the initial stiffness of a given solid. For example, the slope for steel is at least 1000 times steeper than the initial slope for rubber, and 100 times steeper than the initial slope for tendons (one of the stiffest biological soft tissue, made of more than 80% collagen in mass). For this reason, rubbers and tissues are often called “soft” solids.

These different behaviors can be explained by different *microscopic* structures. Hence a typical metal has an atomic lattice structure, with movements governed by short-range strong forces. A typical rubber-like material is made up of long chain molecules, which are spread randomly and can move quite loosely one with respect to another, with a few points of contact. Biological soft tissues are essentially made of an elastin matrix with embedded collagen fibers. (The latter are three orders of magnitude stiffer than the former, but are crumpled in an undeformed tissue; their contribution is felt in the strain-stiffening part of the curve.) See Figure 2 for a schematization of these microscopic structures.

We conclude the introduction with a mention of some non-elastic behaviours, not treated here, such as fracture, viscoelasticity, plasticity, or damage.

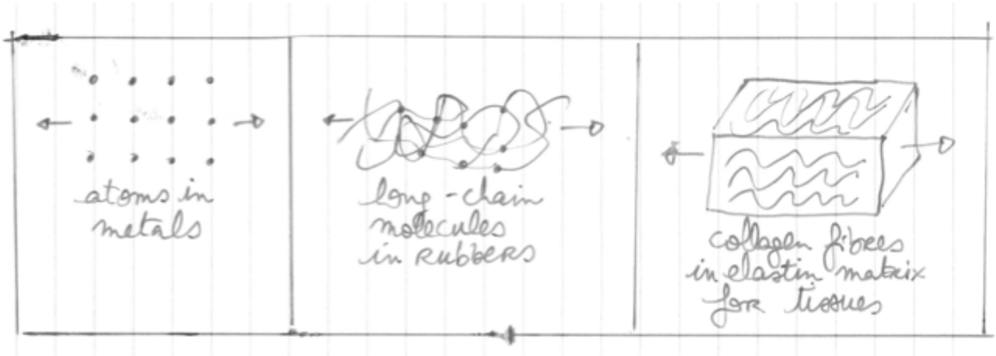

Figure 2: Sketch of micro-structure for (a) a metal, (b) a rubber, (c) a soft tissue.

1.2 Some algebra

A square $n \times n$ matrix $\boldsymbol{A}$ is an array of n^2 elements, the *components* of $\boldsymbol{A}$:

$$\boldsymbol{A} = (A_{ij}) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}. \tag{1.1}$$

In $\boldsymbol{A}^T$, the *transpose* of $\boldsymbol{A}$, rows become columns and vice-versa:

$$\boldsymbol{A}^T = (A_{ij})^T = (A_{ji}) = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \dots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}. \tag{1.2}$$

Symmetric matrix: $\boldsymbol{A} = \boldsymbol{A}^T$, $A_{ij} = A_{ji}$,
Antisymmetric matrix: $\boldsymbol{A} = -\boldsymbol{A}^T$, $A_{ij} = -A_{ji}$.

The 3×3 *unit matrix* is

$$\boldsymbol{I} = (\delta_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.3}$$

Hence, $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$. This is the so-called *Kronecker delta.* It gives

$$\sum_{j=1}^{3} \delta_{ij} A_{jk} = A_{ik}. \tag{1.4}$$

The *trace* of a square matrix is the sum of the terms on its leading diagonal:

$$\operatorname{tr}(\boldsymbol{A}) = A_{11} + A_{22} + \dots A_{nn}. \tag{1.5}$$

In particular, $\operatorname{tr}(\boldsymbol{I}) = 3$.

The *determinant* of a 3×3 matrix is defined as

$$\det(\boldsymbol{A}) = \frac{1}{6} \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{p=1}^{3} \sum_{q=1}^{3} \sum_{r=1}^{3} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} A_{kr}, \tag{1.6}$$

where ϵ_{ijk} is the *alternating symbol*, which is equal to $+1$ when ijk is an even permutation, to -1 when ijk is an odd permutation, and to 0 when i, j, and/or k is/are repeated:

$$\begin{cases} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1, \\ \epsilon_{132} = \epsilon_{312} = \epsilon_{213} = -1, \\ \epsilon_{ijk} = 0 \text{ when an index is repeated.} \end{cases}$$

So, although there are $3^6 = 729$ terms to be summed in the definition of $\det \boldsymbol{A}$, only 6 are non-zero. Important properties of the determinant:

$$\det(\boldsymbol{A}^T) = \det \boldsymbol{A}, \qquad \det(\boldsymbol{A}\boldsymbol{B}) = (\det \boldsymbol{A})(\det \boldsymbol{B}), \qquad \det(\boldsymbol{I}) = 1. \tag{1.7}$$

A matrix $\boldsymbol{A}$ has an *inverse* $\boldsymbol{A}^{-1}$ if and only if $\det(\boldsymbol{A}) \neq 0$.

An *orthogonal matrix* $\boldsymbol{Q}$ is such that $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$. It then follows that (why?) that $\det \boldsymbol{Q} = \pm 1$. A *proper orthogonal matrix*, also known as a *rotation*, is such that

$$\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}, \qquad \det(\boldsymbol{Q}) = +1. \tag{1.8}$$

The *summation convention*, or *Einstein convention*, is a most useful shortcut notation: if the same index occurs twice, the summation from 1 to 3 is assumed. For example, the trace and the determinant can be written, respectively, as

$$\operatorname{tr}(\boldsymbol{A}) = A_{11} + A_{22} + A_{33} = A_{ii}, \qquad \det(\boldsymbol{A}) = \tfrac{1}{6} \epsilon_{ijk} \epsilon_{pqr} A_{ip} A_{jq} A_{kr}. \tag{1.9}$$

The *dot product* of two vectors

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{is} \quad \boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = u_i v_i, \tag{1.10}$$

and the *cross product* of two vectors

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}, \text{ has component } (\boldsymbol{u} \times \boldsymbol{v})_i = \epsilon_{ijk} u_j v_k. \tag{1.11}$$

Now we define the *magnitude* $|\boldsymbol{a}|$ of a vector $\boldsymbol{a}$, as

$$|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}} = \sqrt{a_i a_i}. \tag{1.12}$$

Recall also the definitions and geometrical interpretations of the *dot product* $\boldsymbol{a} \cdot \boldsymbol{b}$ and the *vector product* $\boldsymbol{a} \times \boldsymbol{b}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos(\boldsymbol{a}, \boldsymbol{b}), \qquad |\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}| \sin(\boldsymbol{a}, \boldsymbol{b}). \tag{1.13}$$

It follows that $|\boldsymbol{a} \times \boldsymbol{b}|$ is the area of the parallelogram with edges $\boldsymbol{a}$ and $\boldsymbol{b}$, and that the *triple product* $(\boldsymbol{a} \times \boldsymbol{b})\boldsymbol{\cdot}\boldsymbol{c}$ is the volume of the parallelepiped with edges $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$.

The *gradient of a scalar* ϕ is a vector. In rectangular coordinate system:

$$\text{grad } \phi = \begin{bmatrix} \dfrac{\partial \phi}{\partial x_1} \\ \dfrac{\partial \phi}{\partial x_2} \\ \dfrac{\partial \phi}{\partial x_3} \end{bmatrix}, \quad \text{with component } \boxed{(\text{grad } \phi)_i = \frac{\partial \phi}{\partial x_j}}. \tag{1.14}$$

The *gradient of a vector* $\boldsymbol{u}$ is a tensor. In rectangular coordinate system:

$$\text{grad } \boldsymbol{u} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}, \quad \text{with component } \boxed{(\text{grad } \boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j}}. \tag{1.15}$$

The *curl of a vector* $\boldsymbol{u}$ is a vector. In rectangular coordinate system:

$$\text{curl } \boldsymbol{u} = \begin{bmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_1} \\ -\dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u_1}{\partial x_2} \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \end{bmatrix}, \quad \text{with component } \boxed{\epsilon_{ijk} \frac{\partial u_j}{\partial x_k}}. \tag{1.16}$$

The *divergence* of a vector is a scalar. In rectangular coordinate system:

$$\text{div } \boldsymbol{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}, \qquad \text{i.e. } \boxed{\text{div } \boldsymbol{u} = \frac{\partial u_i}{\partial x_i}}. \tag{1.17}$$

The *divergence* of a tensor is a vector. In rectangular coordinate system:

$$\text{div } \boldsymbol{T} = \begin{bmatrix} \dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial T_{21}}{\partial x_2} + \dfrac{\partial T_{31}}{\partial x_3} \\ \dfrac{\partial T_{12}}{\partial x_1} + \dfrac{\partial T_{22}}{\partial x_2} + \dfrac{\partial T_{32}}{\partial x_3} \\ \dfrac{\partial T_{13}}{\partial x_1} + \dfrac{\partial T_{23}}{\partial x_2} + \dfrac{\partial T_{33}}{\partial x_3} \end{bmatrix}, \qquad \text{with component } \boxed{(\text{div } \boldsymbol{T})_i = \frac{\partial T_{ji}}{\partial x_j}}. \tag{1.18}$$

(note that this is the convention used throughout this course. Some books do the sum on the second index instead).

2 Kinematics

2.1 Bodies, configurations and motions

Definition

A *body* $\mathcal{B}$ is a continuous set whose elements can be put into one-to-one correspondence with points of a region B in three-dimensional Euclidean point space. The elements of $\mathcal{B}$ are called *particles* (or *material points*) and B is called a *configuration* of $\mathcal{B}$.

As the body moves, the configuration changes with time. Let $t \in I \subset \mathbb{R}$ denote time, where I is an interval in $\mathbb{R}$. If, with each $t \in I$, we associate a unique configuration B_t of $\mathcal{B}$ then the family of configurations $\{\mathrm{B}_t : t \in I\}$ is called a *motion* of $\mathcal{B}$. We assume that as $\mathcal{B}$ moves continuously then B_t changes continuously.

It is convenient to identify a *reference configuration*, B_r say, which is an arbitrarily chosen fixed configuration. Then, any particle P of the body $\mathcal{B}$ may be labelled by its position vector $\boldsymbol{X}$ in B_r relative to some origin O. Let $\boldsymbol{x}$ be the position vector of P in the configuration B_t at time t relative to an origin o (which need not coincide with O), as depicted in Figure 3.

We say that the body $\mathcal{B}$ *occupies* the configuration B_t at time t — B_t is also referred to as the *current configuration*. Note that B_r need not be a configuration actually occupied by $\mathcal{B}$ during the motion but that it is often chosen to be the configuration occupied by $\mathcal{B}$ at some prescribed (initial) time.

Since B_r and B_t are configurations of $\mathcal{B}$, there exists a bijection mapping $\boldsymbol{\chi}$: $\mathrm{B}_r \to \mathrm{B}_t$ such that

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}) \quad \text{for all } \boldsymbol{X} \in \mathrm{B}_r, \qquad \boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \mathrm{B}_t. \tag{2.1}$$

The mapping $\boldsymbol{\chi}$ is called the *deformation* of the body *from* B_r *to* B_t. Since B_t depends on t we write

$$\boxed{\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)} \qquad \text{for all } \boldsymbol{X} \in \mathrm{B}_r,\ t \in I. \tag{2.2}$$

For each particle P (with label $\boldsymbol{X}$) this describes the motion of P with t as parameter, and hence the motion of the whole body $\mathcal{B}$. It is usual to assume that $\boldsymbol{\chi}(\boldsymbol{X}, t)$ is twice-continuously differentiable with respect to position and time, although there are situations where this requirement needs to be relaxed (For example, across a phase boundary where one or more of the first or second derivatives of $\boldsymbol{\chi}$ is discontinuous.)

Example: **Rigid motion**

A motion is said to be *rigid* if the distance between any two particles of $\mathcal{B}$ does not change during the motion.

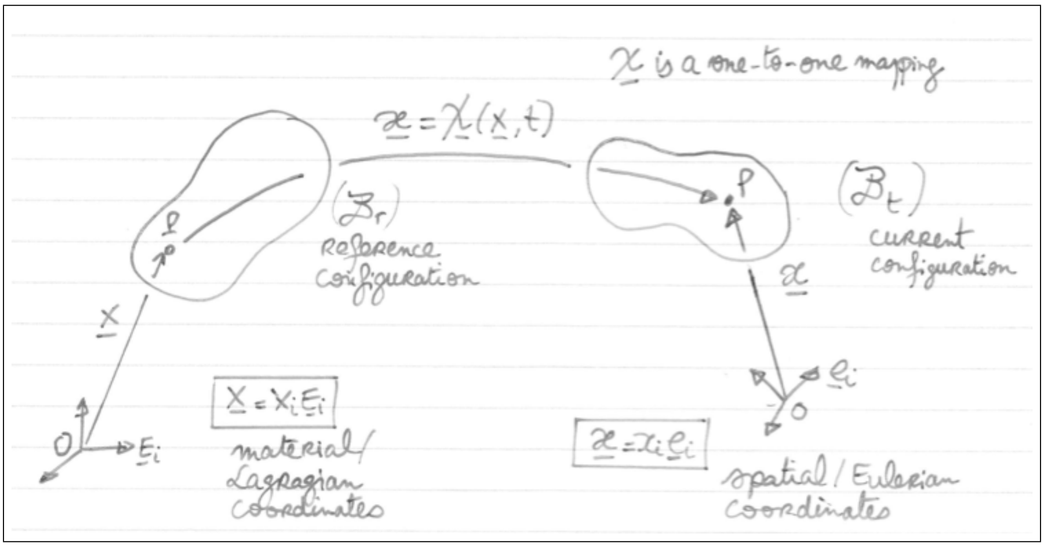

Figure 3: Reference configuration B_r and current configuration B_t with position vectors $\boldsymbol{X}$ and $\boldsymbol{x}$ of a material point P.

The motion defined by

$$\boldsymbol{x} \equiv \boldsymbol{\chi}(\boldsymbol{X}, t) = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{X}, \tag{2.3}$$

where $\boldsymbol{c}(t)$ is a vector and $\boldsymbol{Q}(t)$ is a proper orthogonal second-order tensor, is a rigid motion. To show this we consider $\boldsymbol{Y} \in \mathrm{B}_r$ so that $\boldsymbol{y} = \boldsymbol{c}(t) + \boldsymbol{Q}(t)\boldsymbol{Y}$. Then

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{y}|^2 &= (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \\ &= [\boldsymbol{Q}(\boldsymbol{X} - \boldsymbol{Y})] \cdot [\boldsymbol{Q}(\boldsymbol{X} - \boldsymbol{Y})] \\ &= [\boldsymbol{Q}^T\boldsymbol{Q}(\boldsymbol{X} - \boldsymbol{Y})] \cdot (\boldsymbol{X} - \boldsymbol{Y}) \\ &= (\boldsymbol{X} - \boldsymbol{Y}) \cdot (\boldsymbol{X} - \boldsymbol{Y}) \\ &= |\boldsymbol{X} - \boldsymbol{Y}|^2, \end{aligned} \tag{2.4}$$

where we have used $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$. In fact, although we have not proved it here, *every* rigid motion can be expressed in the form (2.3). Note that $\boldsymbol{c}(t)$ represents a *translation* and $\boldsymbol{Q}(t)$ a *rotation.*

In the development of the basic principles of continuum mechanics, a body $\mathcal{B}$ is endowed with various physical properties which are represented by scalar, vector and tensor fields defined on *either* B_r *or* B_t (for example, density, temperature, shape of surface, velocity, strain). In the case of B_r the position vector $\boldsymbol{X}$ and time t serve as independent variables, and the fields are then said to be defined in terms of the *referential* or *material* description. Alternatively, in the case of B_t, $\boldsymbol{x}$ and t are used and the description is said to be *spatial.* The terminologies *Lagrangian* and *Eulerian descriptions* are also used in respect of B_r and B_t, respectively.

Rectangular Cartesian coordinate systems with basis vectors $\{\boldsymbol{E}_i\}$ and $\{\boldsymbol{e}_i\}$ are chosen for B_r and B_t respectively, with *material coordinates* X_i and *spatial coordi-*

nates x_i $(i = 1, 2, 3)$. Thus, relative to the origins O and o respectively, we have

$$\boldsymbol{X} = X_i \boldsymbol{E}_i, \qquad \boldsymbol{x} = x_i \boldsymbol{e}_i. \tag{2.5}$$

In (2.5) the summation convention over repeated indices applies. It will also apply henceforth except where stated otherwise. In general, $\boldsymbol{E}_i$ and $\boldsymbol{e}_i$ may be chosen to have different orientations, but it is often convenient to let them coincide.

2.2 The material time derivative

The *velocity* $\boldsymbol{v}$ of a particle P is defined as

$$\boldsymbol{v} \equiv \dot{\boldsymbol{x}} = \frac{\partial}{\partial t} \boldsymbol{\chi}(\boldsymbol{X}, t), \tag{2.6}$$

i.e. the rate of change of position of P (or $\partial/\partial t$ at fixed $\boldsymbol{X}$). The *acceleration* $\boldsymbol{a}$ of P is

$$\boldsymbol{a} \equiv \dot{\boldsymbol{v}} \equiv \ddot{\boldsymbol{x}} = \frac{\partial^2}{\partial t^2} \boldsymbol{\chi}(\boldsymbol{X}, t). \tag{2.7}$$

In each case a superposed dot indicates differentiation with respect to t at fixed $\boldsymbol{X}$.

Let ϕ be a scalar field defined on B_t, i.e. $\phi = \phi(\boldsymbol{x}, t)$ is the *spatial description* of the field. Since $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, we may write that

$$\phi(\boldsymbol{x}, t) = \phi[\boldsymbol{\chi}(\boldsymbol{X}, t), t] \equiv \Phi(\boldsymbol{X}, t), \tag{2.8}$$

which defines the notation Φ, the *material description* of the field. Thus, any field defined on B_t (respectively B_r) can, through (2.2) or its inverse, equally be defined on B_r (respectively B_t).

The *material derivative* of ϕ is the rate of change of ϕ at fixed *material point* P, i.e. at fixed $\boldsymbol{X}$. We write the material derivative as $\dot{\phi}$ or $D\phi/Dt$.

By definition, we have

$$\dot{\phi} = \frac{\partial}{\partial t} \Phi(\boldsymbol{X}, t), \tag{2.9}$$

and by the chain rule for partial derivatives we then obtain

$$\frac{\partial}{\partial t} \Phi(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{x}}{\partial t} \frac{\partial}{\partial \boldsymbol{x}} \phi(\boldsymbol{x}, t) + \frac{\partial}{\partial t} \phi(\boldsymbol{x}, t) = \frac{\partial}{\partial t} \phi(\boldsymbol{x}, t) + \frac{\partial \boldsymbol{x}}{\partial t} \cdot \operatorname{grad} \phi(\boldsymbol{x}, t), \tag{2.10}$$

where grad denotes the gradient operator with respect to $\boldsymbol{x}$. Using (2.6) we thus have

$$\boxed{\underbrace{\frac{\partial}{\partial t} \Phi(\boldsymbol{X}, t)}_{\text{material description}} \equiv \dot{\phi} = \underbrace{\frac{\partial}{\partial t} \phi + \boldsymbol{v} \cdot \operatorname{grad} \phi}_{\text{spatial description}},} \tag{2.11}$$

or

$$\frac{\partial \Phi}{\partial t} = \dot{\phi} = \frac{\partial \phi}{\partial t} + v_j \frac{\partial \phi}{\partial x_j}. \tag{2.12}$$

We can proceed similarly for a vector field. First we define the *spatial* ($\boldsymbol{u}$) and *material* ($\boldsymbol{U}$) descriptions of the vector field:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}[\boldsymbol{\chi}(\boldsymbol{X}, t), t] = \boldsymbol{U}(\boldsymbol{X}, t), \tag{2.13}$$

or, in components:

$$u_i(\boldsymbol{x}, t) = u_i[\boldsymbol{\chi}(\boldsymbol{X}, t), t] = U_i(\boldsymbol{X}, t). \tag{2.14}$$

Then, similarly to (2.12), we can compute the material time derivative of the $i-$th component of the vector field:

$$\frac{\partial U_i}{\partial t} = \dot{u}_i = \frac{\partial u_i}{\partial t} + v_j \frac{\partial u_i}{\partial x_j}. \tag{2.15}$$

Here we notice that $\partial u_i / \partial x_j$ is the ij-th component of the tensor $\operatorname{grad} \boldsymbol{u}$, and that the last term is the product of a tensor by a vector. This equation can thus be written as

$$\boxed{\frac{\partial}{\partial t} \boldsymbol{U}(\boldsymbol{X}, t) \equiv \dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{u}}{\partial t} + (\operatorname{grad} \boldsymbol{u})\boldsymbol{v},} \tag{2.16}$$

In particular, the acceleration $\boldsymbol{a} = \dot{\boldsymbol{v}}$ is given by

$$\boxed{\boldsymbol{a} = \dot{\boldsymbol{v}} = \frac{\partial \boldsymbol{v}}{\partial t} + (\operatorname{grad} \boldsymbol{v})\boldsymbol{v}.} \tag{2.17}$$

2.3 Differentiation of Cartesian tensor fields

Let ϕ, $\boldsymbol{u}$, $\boldsymbol{T}$ be scalar, vector and tensor functions of position $\boldsymbol{x}$, respectively. Recall that the operation of the *gradient* operator, grad, on ϕ and $\boldsymbol{u}$ with respect to the basis $\{\boldsymbol{e}_i\}$ is defined as follows:

$$(\operatorname{grad} \phi)_i = \frac{\partial \phi}{\partial x_i} \tag{2.18}$$

$$(\operatorname{grad} \boldsymbol{u})_{ij} = \frac{\partial u_i}{\partial x_j}. \tag{2.19}$$

Note that the operation of grad increases the order of the tensor by one. On the other hand, the operation of the *divergence* operator, div, reduces the order of a tensor by one. The divergence of a vector is a scalar and the divergence of a tensor is a vector:

$$\operatorname{div} \boldsymbol{u} = \frac{\partial u_i}{\partial x_i}, \tag{2.20}$$

$$(\operatorname{div} \boldsymbol{T})_i = \frac{\partial T_{ji}}{\partial x_j}. \tag{2.21}$$

The later equation *defines* the divergence of a second-order tensor for this course (Note that some textbooks define it as $\frac{\partial T_{ij}}{\partial x_j} \boldsymbol{e}_i$.)

2.4 Deformation gradient

Let Grad, Div, Curl (respectively grad, div, curl) denote the gradient, divergence and curl operators in the reference (respectively current) configuration, i.e. with respect to $\boldsymbol{X}$ (respectively $\boldsymbol{x}$). Then, we define the *deformation gradient tensor* $\boldsymbol{F}$ as

$$\boldsymbol{F}(\boldsymbol{X}, t) = \mathrm{Grad}\, \boldsymbol{x} \equiv \mathrm{Grad}\, \boldsymbol{\chi}(\boldsymbol{X}, t). \tag{2.22}$$

With respect to the chosen basis vectors and with use of (2.19) we have in component form,

$$\boxed{F_{ij} = \frac{\partial x_i}{\partial X_j},} \tag{2.23}$$

with $x_i = \chi_i(\boldsymbol{X}, t)$.

We assume that $\det \boldsymbol{F} \neq 0$ (to be justified shortly) so that $\boldsymbol{F}$ has an inverse $\boldsymbol{F}^{-1}$, given by

$$\boldsymbol{F}^{-1} = \mathrm{grad}\, \boldsymbol{X}, \tag{2.24}$$

with components

$$(\boldsymbol{F}^{-1})_{ij} = \frac{\partial X_i}{\partial x_j}. \tag{2.25}$$

This may be checked by means of the calculation

$$(\boldsymbol{F}\boldsymbol{F}^{-1})_{ij} = F_{ik}(\boldsymbol{F}^{-1})_{kj} = \frac{\partial x_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij}. \tag{2.26}$$

It follows from (2.23) that

$$F_{ij} dX_j = \frac{\partial x_i}{\partial X_j} dX_j = dx_i, \tag{2.27}$$

i.e.

$$d\boldsymbol{x} = \boldsymbol{F} d\boldsymbol{X}, \tag{2.28}$$

which has inverse

$$d\boldsymbol{X} = \boldsymbol{F}^{-1} d\boldsymbol{x}. \tag{2.29}$$

Equation (2.28) describes how infinitesimal *line elements* $d\boldsymbol{X}$ of material at $\boldsymbol{X}$ transform under the deformation into $d\boldsymbol{x}$ (which consists of the same material as $d\boldsymbol{X}$) at $\boldsymbol{x}$. It shows that locally, *line elements* transform *linearly* since $\boldsymbol{F}$ depends on $\boldsymbol{X}$ (and not on $d\boldsymbol{X}$). Thus, at each $\boldsymbol{X}$, $\boldsymbol{F}$ is a *linear mapping* (i.e. a second-order tensor).

We justify taking $\boldsymbol{F}$ to be *non-singular* ($\det \boldsymbol{F} \neq 0$) by noting that $\boldsymbol{F} d\boldsymbol{X} \neq \boldsymbol{0}$ if $d\boldsymbol{X} \neq \boldsymbol{0}$, i.e. a line element cannot be annihilated by the deformation process.

Examples

Let ϕ, $\boldsymbol{u}$, $\boldsymbol{T}$ respectively be scalar, vector, and second-order tensor fields associated with a moving body. The following formulas will prove very useful:

$$\operatorname{Grad}\phi = \boldsymbol{F}^T \operatorname{grad}\phi, \tag{2.30}$$

$$\operatorname{Grad}\boldsymbol{u} = (\operatorname{grad}\boldsymbol{u})\boldsymbol{F}, \tag{2.31}$$

$$\operatorname{Div}\boldsymbol{u} = J\operatorname{div}(J^{-1}\boldsymbol{F}\boldsymbol{u}), \tag{2.32}$$

$$\operatorname{Div}\boldsymbol{T} = J\operatorname{div}(J^{-1}\boldsymbol{F}\boldsymbol{T}), \tag{2.33}$$

where J is defined as

$$J = \det \boldsymbol{F}. \tag{2.34}$$

To show the first formula (2.30), we calculate

$$\begin{aligned}\left(\boldsymbol{F}^T \operatorname{grad}\phi\right)_i &= F^T_{ij}\left(\operatorname{grad}\phi\right)_j = F_{ji}\frac{\partial \phi}{\partial x_j} = F_{ji}\frac{\partial X_k}{\partial x_j}\frac{\partial \phi}{\partial X_k} \\ &= F_{ji}F^{-1}_{kj}\frac{\partial \phi}{\partial X_k} = \delta_{ki}\frac{\partial \phi}{\partial X_k} = \frac{\partial \phi}{\partial X_i} = \left(\operatorname{Grad}\phi\right)_i.\end{aligned} \tag{2.35}$$

The second formula is proved in the same way. The third and fourth formulas require more work and are proved later (Section 2.10).

2.5 Deformation of line, area and volume elements

We saw that equation (2.28) describes how infinitesimal *line elements* $d\boldsymbol{X}$ of material initially at $\boldsymbol{X}$ transform under the deformation into infinitesimal line elements $d\boldsymbol{x}$ currently at $\boldsymbol{x}$:

$$\boxed{d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}.} \tag{2.36}$$

Now, consider the parallelepiped in B_r formed by line elements $d\boldsymbol{X}$, $d\boldsymbol{X}'$, $d\boldsymbol{X}''$ at $\boldsymbol{X}$. Its volume dV is given by

$$dV = d\boldsymbol{X}\cdot(d\boldsymbol{X}'\times d\boldsymbol{X}'') = \det\left(d\boldsymbol{X}\middle|d\boldsymbol{X}'\middle|d\boldsymbol{X}''\right). \tag{2.37}$$

The corresponding volume dv in B_t is

$$\begin{aligned}dv = d\boldsymbol{x}\cdot(d\boldsymbol{x}'\times d\boldsymbol{x}'') &= \det\left(d\boldsymbol{x}\middle|d\boldsymbol{x}'\middle|d\boldsymbol{x}''\right) = \det\left(\boldsymbol{F}d\boldsymbol{X}\middle|\boldsymbol{F}d\boldsymbol{X}'\middle|\boldsymbol{F}d\boldsymbol{X}''\right) \\ &= \det\left[\boldsymbol{F}\left(d\boldsymbol{X}\middle|d\boldsymbol{X}'\middle|d\boldsymbol{X}''\right)\right] = \det(\boldsymbol{F})\det\left(d\boldsymbol{X}\middle|d\boldsymbol{X}'\middle|d\boldsymbol{X}''\right),\end{aligned}$$

i.e.

$$\boxed{dv = JdV.} \tag{2.38}$$

Recalling that $\boldsymbol{F}$ is nonsingular, it is appropriate, by convention, to define volume elements to be positive, so that

$$J \equiv \det\boldsymbol{F} > 0. \tag{2.39}$$

From (2.38) we see that J is a measure of the *change in volume* under the deformation. If the deformation is such that there is no change in volume then the deformation is said to be *isochoric*, and then

$$J \equiv \det \boldsymbol{F} = 1. \tag{2.40}$$

For some materials many deformations are such that (2.40) holds to a good approximation, and (2.40) is adopted as an *idealization.* An (ideal) material for which (2.40) holds for *all* deformations is called an *incompressible material.*

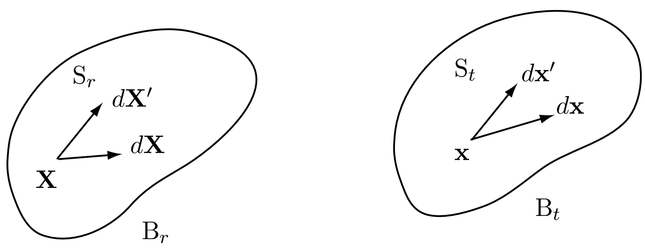

Figure 4: Infinitesimal line elements at $\boldsymbol{X}$ on the surface S_r in the reference configuration B_r and their images at $\boldsymbol{x}$ on the deformed surface S_t in the current configuration B_t.

Next, consider a surface S_r in the reference configuration B_r which deforms into the surface S_t in the current configuration B_t, as depicted in Figure 4. Let $\boldsymbol{X}$ be a point on S_r and $\boldsymbol{x}$ be the corresponding point on S_t. Let $d\boldsymbol{X}$ and $d\boldsymbol{X}'$ be infinitesimal line elements of material on S_r based at $\boldsymbol{X}$ with images $d\boldsymbol{x}$ and $d\boldsymbol{x}'$ on S_t under the deformation. Strictly, the line elements are tangential to the surface and only approximately lie *in* the surface. If $\boldsymbol{F}$ denotes the deformation gradient, then

$$d\boldsymbol{x} = \boldsymbol{F} d\boldsymbol{X}, \qquad d\boldsymbol{x}' = \boldsymbol{F} d\boldsymbol{X}'. \tag{2.41}$$

Let dA and da be surface area elements on S_r and S_t respectively, and let $\boldsymbol{N}$ and $\boldsymbol{n}$ be unit normals at $\boldsymbol{X}$ and $\boldsymbol{x}$ respectively. For the parallelogram with sides $d\boldsymbol{X}$, $d\boldsymbol{X}'$ we have

$$\boldsymbol{N} dA = d\boldsymbol{X} \times d\boldsymbol{X}'. \tag{2.42}$$

Under the deformation this becomes a parallelogram with sides $d\boldsymbol{x}$, $d\boldsymbol{x}'$ and area

$$\boldsymbol{n} da = d\boldsymbol{x} \times d\boldsymbol{x}'. \tag{2.43}$$

From (2.38) we obtain, for any third line element $d\boldsymbol{x}'' = \boldsymbol{F} d\boldsymbol{X}''$,

$$d\boldsymbol{x}'' \cdot \boldsymbol{n} da = J d\boldsymbol{X}'' \cdot \boldsymbol{N} dA = J \boldsymbol{F}^{-1} d\boldsymbol{x}'' \cdot \boldsymbol{N} dA = J d\boldsymbol{x}'' \cdot \left(\boldsymbol{F}^{-1}\right)^T \boldsymbol{N} dA. \tag{2.44}$$

Hence

$$\boldsymbol{n} da = J (\boldsymbol{F}^{-1})^T \boldsymbol{N} dA, \tag{2.45}$$

where $J = \det \boldsymbol{F}$. With the notation

$$\boldsymbol{F}^{-T} = (\boldsymbol{F}^{-1})^T = (\boldsymbol{F}^T)^{-1}, \tag{2.46}$$

this becomes

$$\boxed{\boldsymbol{n} da = J\boldsymbol{F}^{-T}\boldsymbol{N} dA.} \tag{2.47}$$

This is an important result, known as *Nanson's formula*, and it describes how elements of *surface area* deform. It applies to area elements of arbitrary shape, not just the parallelogram considered here.

In order to analyze further the local nature of the deformation, i.e. of $\boldsymbol{F}$, we require some properties of second-order tensors.

2.6 Some results from tensor algebra

2.6.1 Eigenvalues and eigenvectors

Let $\boldsymbol{S}$ be a real symmetric second-order tensor: $\bar{\boldsymbol{S}} = \boldsymbol{S} = \boldsymbol{S}^T$. The *eigenvalues* of $\boldsymbol{S}$ are the roots λ_1, λ_2, λ_3 of the cubic

$$\det(\boldsymbol{S} - \lambda \boldsymbol{I}) = 0, \tag{2.48}$$

called the *characteristic equation* of $\boldsymbol{S}$. It can be expanded as

$$\boxed{-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0,} \tag{2.49}$$

where the quantities I_1, I_2, I_3 are called the first three *principal invariants* of $\boldsymbol{S}$, and are given by

$$\boxed{I_1 = \operatorname{tr}\boldsymbol{S}, \qquad I_2 = \tfrac{1}{2}\left[(\operatorname{tr}\boldsymbol{S})^2 - \operatorname{tr}\left(\boldsymbol{S}^2\right)\right], \qquad I_3 = \det \boldsymbol{S}.} \tag{2.50}$$

Note that the cubic is also: $-(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = 0$, so that the following identifications apply:

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3, \qquad I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \qquad I_3 = \lambda_1\lambda_2\lambda_3. \tag{2.51}$$

By solving the vector equation $\boldsymbol{S}\boldsymbol{u} = \lambda_1\boldsymbol{u}$ (i.e. $(\boldsymbol{S} - \lambda_1\boldsymbol{I})\,\boldsymbol{u} = \boldsymbol{0}$) for $\boldsymbol{u}$, we find the *eigenvectors* of $\boldsymbol{S}$ corresponding to λ_1. They are all scalar multiples one of another. We call $\boldsymbol{r}$ the unit eigenvector ($\boldsymbol{r} \cdot \boldsymbol{r} = 1$) corresponding to λ_1. Similarly, we define $\boldsymbol{s}$ and $\boldsymbol{t}$ such that

$$\boldsymbol{S}\boldsymbol{r} = \lambda_1\boldsymbol{r}, \qquad \boldsymbol{S}\boldsymbol{s} = \lambda_2\boldsymbol{s}, \qquad \boldsymbol{S}\boldsymbol{t} = \lambda_3\boldsymbol{t}, \tag{2.52}$$

and $(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t})$ is a direct orthonormal basis. Here we need to prove that the eigenvectors corresponding to distinct eigenvalues are indeed orthogonal. Simply write the scalar product $\boldsymbol{r} \cdot \boldsymbol{S}\boldsymbol{s}$:

$$\boldsymbol{r} \cdot \boldsymbol{S}\boldsymbol{s} = \boldsymbol{r}\cdot(\lambda_2\boldsymbol{s}) = \lambda_2\boldsymbol{r} \cdot \boldsymbol{s}. \tag{2.53}$$

On the other hand, it is also equal to

$$\boldsymbol{s} \cdot \boldsymbol{S}^T \boldsymbol{r} = \boldsymbol{s} \cdot \boldsymbol{S}\boldsymbol{r} = \boldsymbol{s} \cdot (\lambda_1 \boldsymbol{r}) = \lambda_1 \boldsymbol{s} \cdot \boldsymbol{r}. \tag{2.54}$$

Hence $(\lambda_2 - \lambda_1)\boldsymbol{r} \cdot \boldsymbol{s} = 0$, i.e. $\boldsymbol{r} \cdot \boldsymbol{s} = 0$.

Similarly, we can prove that the eigenvalues of the real symmetric tensor $\boldsymbol{S}$ are *real*, by taking the complex conjugate of $\bar{\boldsymbol{r}} \cdot \boldsymbol{S}\boldsymbol{r}$:

$$\overline{\bar{\boldsymbol{r}} \cdot \boldsymbol{S}\boldsymbol{r}} = \boldsymbol{r} \cdot \boldsymbol{S}\bar{\boldsymbol{r}} = \bar{\boldsymbol{r}} \cdot \boldsymbol{S}^T \boldsymbol{r} = \bar{\boldsymbol{r}} \cdot \boldsymbol{S}\boldsymbol{r}. \tag{2.55}$$

Hence $\bar{\boldsymbol{r}} \cdot \boldsymbol{S}\boldsymbol{r}$ is equal to its complex conjugate and is thus real. Since it is equal to $\lambda_1 \boldsymbol{r} \cdot \bar{\boldsymbol{r}}$, it follows that λ_1 is real. Note that therefore, $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$ are also real because they are the unit eigenvectors of an eigensystem with real coefficients.

Finally, note that these results were established by assuming that the cubic (2.49) yields three distinct roots. In the case of one double root plus a single root, or of a triple root, some adjustments have to be made, but the main results remain: we can always find a direct orthonormal basis made of real unit eigenvectors of $\boldsymbol{S}$.

2.6.2 The Cayley-Hamilton theorem

Any vector $\boldsymbol{u}$ can be decomposed in the orthonormal basis $(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t})$, as $\boldsymbol{u} = \alpha\boldsymbol{r} + \beta\boldsymbol{s} + \gamma\boldsymbol{t}$, say, for some scalars α, β, γ. Now consider the following operation:

$$\begin{aligned}
&-(\boldsymbol{S} - \lambda_1\boldsymbol{I})(\boldsymbol{S} - \lambda_2\boldsymbol{I})(\boldsymbol{S} - \lambda_3\boldsymbol{I})\,\boldsymbol{u} \\
&\qquad = -(\boldsymbol{S} - \lambda_1\boldsymbol{I})(\boldsymbol{S} - \lambda_2\boldsymbol{I})\left[\alpha(\lambda_1 - \lambda_3)\boldsymbol{r} + \beta(\lambda_2 - \lambda_3)\boldsymbol{s}\right] \\
&\qquad\qquad = -(\boldsymbol{S} - \lambda_1\boldsymbol{I})\left[\alpha(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_2)\boldsymbol{r}\right] = \boldsymbol{0},
\end{aligned} \tag{2.56}$$

for all $\boldsymbol{u}$. Hence $-(\boldsymbol{S} - \lambda_1\boldsymbol{I})(\boldsymbol{S} - \lambda_2\boldsymbol{I})(\boldsymbol{S} - \lambda_3\boldsymbol{I}) = \boldsymbol{O}$, or, by expanding, and using (2.51),

$$\boxed{-\boldsymbol{S}^3 + I_1\boldsymbol{S}^2 - I_2\boldsymbol{S} + I_3\boldsymbol{I} = \boldsymbol{O}.} \tag{2.57}$$

In other words, $\boldsymbol{S}$ satisfies its own characteristic equation.

2.6.3 The spectral decomposition theorem

Let $(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t})$ be any direct orthonormal basis. Consider the second-order tensor $\boldsymbol{A}$ with Cartesian components

$$A_{ij} = r_i r_j + s_i s_j + t_i t_j. \tag{2.58}$$

Any vector $\boldsymbol{u}$ can be written as $\boldsymbol{u} = \alpha\boldsymbol{r} + \beta\boldsymbol{s} + \gamma\boldsymbol{t}$ for some scalars α, β and γ. Computing $\boldsymbol{A}\boldsymbol{u}$ gives $A_{ij}u_j = \alpha r_i + \beta s_i + \gamma t_i = u_i$, where we used $r_j r_j = s_j s_j = t_j t_j = 1$ and $r_j s_j = r_j t_j = s_j t_j = 0$. Hence, $\boldsymbol{A} = \boldsymbol{I}$, the identity:

$$\delta_{ij} = r_i r_j + s_i s_j + t_i t_j. \tag{2.59}$$

Now take the case where ($\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$) are the orthonormal eigenvectors of the symmetric second-order tensor $\boldsymbol{S}$, and consider the components of $\boldsymbol{S} = \boldsymbol{S}\boldsymbol{I}$:

$$S_{ij} = S_{ik}\delta_{kj} = S_{ik}(r_k r_j + s_k s_j + t_k t_j) = \lambda_1 r_i r_j + \lambda_2 s_i s_j + \lambda_3 t_i t_j, \tag{2.60}$$

This is the *spectral decomposition* of $\boldsymbol{S}$. Using it to compute the components of $\boldsymbol{S}$ in the orthonormal basis of its eigenvectors, we find $S_{11} = \lambda_1$, $S_{12} = 0$, etc. In that basis, $\boldsymbol{S}$ has only three non-zero, diagonal, components

$$\boldsymbol{S} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}. \tag{2.61}$$

2.6.4 The square root theorem

A second-order tensor $\boldsymbol{A}$ is *positive definite* when $\boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{u} > 0$, for all non-zero vector $\boldsymbol{u}$.

If $\boldsymbol{S}$ is a positive definite, symmetric second-order tensor, then there exists a unique, positive definite, symmetric second-order tensor, $\boldsymbol{U}$ say, such that

$$\boxed{\boldsymbol{U}^2 = \boldsymbol{S}}. \tag{2.62}$$

Proof Since $\boldsymbol{S}$ is symmetric we may write it in the spectral form (2.61) in the orthonormal basis $(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t})$, where λ_i are the (real) eigenvalues of $\boldsymbol{S}$ and $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$ are the (unit) eigenvectors. Since $\boldsymbol{S}$ is positive definite, we have $\lambda_i > 0$. Now define $\boldsymbol{U}$ by its components in $(\boldsymbol{r}, \boldsymbol{s}, \boldsymbol{t})$:

$$\boldsymbol{U} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \\ 0 & 0 & \sqrt{\lambda_3} \end{bmatrix}. \tag{2.63}$$

Then, $\boldsymbol{U}$ is positive definite and symmetric and $\boldsymbol{U}^2 = \boldsymbol{S}$, as required. Uniqueness is obvious.

2.6.5 The polar decomposition theorem

Let $\boldsymbol{F}$ be a second-order Cartesian tensor such that $\det \boldsymbol{F} > 0$. Then there exist unique, positive definite, symmetric tensors, $\boldsymbol{U}$ and $\boldsymbol{V}$, and a unique proper orthogonal tensor $\boldsymbol{R}$ such that

$$\boxed{\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}.} \tag{2.64}$$

Proof The tensors $\boldsymbol{F}\boldsymbol{F}^T$ and $\boldsymbol{F}^T\boldsymbol{F}$ are symmetric and positive definite (because $\boldsymbol{u} \cdot \boldsymbol{F}^T\boldsymbol{F}\boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{F}^T(\boldsymbol{F}\boldsymbol{u}) = \boldsymbol{F}\boldsymbol{u} \cdot \boldsymbol{F}\boldsymbol{u} = |\boldsymbol{F}\boldsymbol{u}|^2 > 0$). Hence, by the square root theorem, there exist unique positive definite symmetric tensors $\boldsymbol{U}$, $\boldsymbol{V}$ such that

$$\boldsymbol{V}^2 = \boldsymbol{F}\boldsymbol{F}^T, \quad \boldsymbol{U}^2 = \boldsymbol{F}^T\boldsymbol{F}. \tag{2.65}$$

Now define $\boldsymbol{R} = \boldsymbol{F}\boldsymbol{U}^{-1}$. We need to prove that $\boldsymbol{R}$ is proper orthogonal. First, we calculate

$$\boldsymbol{R}^T\boldsymbol{R} = (\boldsymbol{F}\boldsymbol{U}^{-1})^T(\boldsymbol{F}\boldsymbol{U}^{-1}) = \boldsymbol{U}^{-1}\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{U}^{-1} = \boldsymbol{U}^{-1}\boldsymbol{U}^2\boldsymbol{U}^{-1} = \boldsymbol{I}, \tag{2.66}$$

and hence we deduce that $\boldsymbol{R}$ is orthogonal. Second, we calculate

$$\det \boldsymbol{R} = \det(\boldsymbol{F}\boldsymbol{U}^{-1}) = (\det \boldsymbol{F})(\det \boldsymbol{U})^{-1} > 0, \tag{2.67}$$

and it follows that $\boldsymbol{R}$ is *proper* orthogonal.

Since $\boldsymbol{U}$ is unique, $\boldsymbol{R}$ is unique and hence $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$. Similarly, $\boldsymbol{F} = \boldsymbol{V}\boldsymbol{S}$, where $\boldsymbol{S}$ is proper orthogonal. Thus,

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{S} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T\boldsymbol{R} = (\boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T)\boldsymbol{R}, \tag{2.68}$$

where the bracketed tensor is symmetric and the last one is a rotation. By uniqueness it follows that $\boldsymbol{S} = \boldsymbol{R}$ and hence (2.64) holds. Note that $\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T$.

Corollary

If $\boldsymbol{U}$ has eigenvalues λ_i and eigenvectors $\boldsymbol{u}^{(i)}$, $i \in \{1, 2, 3\}$, then $\lambda_i > 0$ and λ_i are also the eigenvalues of $\boldsymbol{V}$ with eigenvectors $\boldsymbol{R}\boldsymbol{u}^{(i)}$.

Proof It follows from symmetry and from positive definiteness of $\boldsymbol{U}$ that $\lambda_i > 0$. Also, we have

$$\boldsymbol{V}(\boldsymbol{R}\boldsymbol{u}^{(1)}) = \boldsymbol{V}\boldsymbol{R}\boldsymbol{u}^{(1)} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{u}^{(1)} = \boldsymbol{R}(\lambda_1\boldsymbol{u}^{(1)}) = \lambda_1(\boldsymbol{R}\boldsymbol{u}^{(1)}), \tag{2.69}$$

which shows that $\boldsymbol{R}\boldsymbol{u}^{(1)}$ is the eigenvector of $\boldsymbol{V}$ associated with λ_1, and similarly for the other eigenvectors.

2.7 Analysis of deformation: Stretch, extension and strain

Let $\boldsymbol{M}$ and $\boldsymbol{m}$ be unit vectors along $d\boldsymbol{X}$ and $d\boldsymbol{x}$ respectively, so that $d\boldsymbol{X} = \boldsymbol{M}|d\boldsymbol{X}|, d\boldsymbol{x} = \boldsymbol{m}|d\boldsymbol{x}|$ and (2.28) gives $\boldsymbol{m}|d\boldsymbol{x}| = \boldsymbol{F}\boldsymbol{M}|d\boldsymbol{X}|$. Thus

$$|d\boldsymbol{x}|^2 = d\,\boldsymbol{x}\cdot d\boldsymbol{x} = (\boldsymbol{F}\boldsymbol{M})\cdot(\boldsymbol{F}\boldsymbol{M})|d\boldsymbol{X}|^2 = (\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{M})\cdot\boldsymbol{M}|d\boldsymbol{X}|^2 \tag{2.70}$$

and hence

$$\frac{|d\boldsymbol{x}|}{|d\boldsymbol{X}|} = |\boldsymbol{F}\boldsymbol{M}| = \sqrt{\boldsymbol{M}\cdot(\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{M})} \equiv \lambda(\boldsymbol{M}), \tag{2.71}$$

which defines $\lambda(\boldsymbol{M})$, called the *stretch in the direction* $\boldsymbol{M}$ at $\boldsymbol{X}$. Note that $0 < \lambda(\boldsymbol{M}) < \infty$ for all unit vectors $\boldsymbol{M}$. The *extension* is the quantity $\lambda - 1$.

Next, from (2.70), we have

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 = d\boldsymbol{X}\cdot(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})d\boldsymbol{X}. \tag{2.72}$$

The material is said to be *unstrained* at $\boldsymbol{X}$ if no line element changes length, i.e.

$$d\boldsymbol{X} \cdot (\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})d\boldsymbol{X} = 0 \quad \text{for all } d\boldsymbol{X}, \tag{2.73}$$

or, equivalently,

$$\lambda(\boldsymbol{M}) = 1 \quad \text{for all } \textit{unit} \text{ vectors } \boldsymbol{M}. \tag{2.74}$$

It follows that $\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I} = \boldsymbol{O}$, the zero tensor. This allows the possibility that $\boldsymbol{F}$ is just a rotation $\boldsymbol{R}$, since, for orthogonal $\boldsymbol{R}$, we have $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$.

Strain is measured locally by changes in the lengths of line elements. In other words, strain measures the changes in distance of two neighboring particles, one at postion $\boldsymbol{X}$, the other at position $\boldsymbol{X} + d\boldsymbol{X}$, in the reference configuration, mapped into positions $\boldsymbol{x}$ and $\boldsymbol{x} + d\boldsymbol{x}$ in the current configuration. For example the value of (2.72) gives the change in the squared length of a line element. Thus, the tensor $\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}$ is a *measure of strain.* The so-called *Green strain tensor* $\boldsymbol{E}$ is defined by

$$\boxed{\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}).} \tag{2.75}$$

Then, by (2.72), the squared lengths of line elements have changed as

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 = 2d\boldsymbol{X} \cdot \boldsymbol{E}d\boldsymbol{X} = 2E_{ij}dX_i dX_j, \tag{2.76}$$

and $\boldsymbol{E}$ refers to changes with respect to the material (Lagrangian) line elements. It is also called the *Lagrangian strain tensor.*

Conversely, we can express the changes with respect to the spatial line elements, as

$$|d\boldsymbol{x}|^2 - |d\boldsymbol{X}|^2 = 2d\boldsymbol{x} \cdot \boldsymbol{e}d\boldsymbol{x}, \quad \text{where} \quad \boldsymbol{e} = \tfrac{1}{2}\left[\boldsymbol{I} - (\boldsymbol{F}^{-1})^T\boldsymbol{F}^{-1}\right] \tag{2.77}$$

is the *Eulerian strain tensor.*

Using the polar decomposition (2.64) for the deformation gradient $\boldsymbol{F}$, we may also form the following tensor measures of deformation:

$$\boxed{\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^2,} \qquad \boxed{\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{V}^2.} \tag{2.78}$$

We refer to $\boldsymbol{C}$ and $\boldsymbol{B}$ as the *right* and *left Cauchy-Green deformation tensors* respectively. Note that the Green strain tensor $\boldsymbol{E}$ may be written as

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \tfrac{1}{2}(\boldsymbol{U}^2 - \boldsymbol{I}), \tag{2.79}$$

and the Eulerian strain tensor $\boldsymbol{e}$ may be written as

$$\boldsymbol{e} = \tfrac{1}{2}(\boldsymbol{I} - \boldsymbol{B}^{-1}) = \tfrac{1}{2}(\boldsymbol{I} - \boldsymbol{V}^{-2}). \tag{2.80}$$

Since $\boldsymbol{U}$ is positive definite and symmetric there exist (unit, orthogonal) eigenvectors $\boldsymbol{u}^{(i)}$ such that its components in the basis $(\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \boldsymbol{u}^{(3)})$ are

$$\boldsymbol{U} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \tag{2.81}$$

where $\lambda_i > 0$ are called the *principal stretches* of the deformation and $\boldsymbol{u}^{(i)}$ are called the *principal directions*. Note that, in accordance with the definition (2.71), $\lambda_i = \lambda(\boldsymbol{u}^{(i)})$ — hence the terminology *principal stretch*.

The tensors $\boldsymbol{U}$ and $\boldsymbol{V}$ are called the *right* and *left stretch tensors* respectively. The deformation $\boldsymbol{F}$ rotates the principal axes of $\boldsymbol{U}$ into those of $\boldsymbol{V}$ as well as stretching along those directions. The principal axes of $\boldsymbol{U}$ and $\boldsymbol{V}$ are sometimes referred to as the *Lagrangian* and *Eulerian principal axes* respectively.

Other strain tensors based on $\boldsymbol{U}$ may be defined. For example, we define $\boldsymbol{E}^{(m)}$ as follows:

$$\boldsymbol{E}^{(m)} = \frac{1}{m}(\boldsymbol{U}^m - \boldsymbol{I}) \quad m \neq 0, \tag{2.82}$$

$$\boldsymbol{E}^{(0)} = \ln \boldsymbol{U}, \tag{2.83}$$

where m is a real number, not necessarily an integer. These are Lagrangian tensors, all coaxial with $\boldsymbol{U}$, and have eigenvalues $(\lambda_i^m - 1)/m$ for $m \neq 0$ and $\ln \lambda_i$ for $m = 0$. Corresponding Eulerian tensors, here denoted $\boldsymbol{e}^{(m)}$ and based on $\boldsymbol{V}$, are defined by

$$\boldsymbol{e}^{(m)} = \frac{1}{m}(\boldsymbol{V}^m - \boldsymbol{I}) \quad m \neq 0, \tag{2.84}$$

$$\boldsymbol{e}^{(0)} = \ln \boldsymbol{V}, \tag{2.85}$$

and we notice that, on recalling the connection $\boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T$, $\boldsymbol{e}^{(m)} = \boldsymbol{R}\boldsymbol{E}^{(m)}\boldsymbol{R}^T$ for each m. Thus, $\boldsymbol{E}^{(m)}$ and $\boldsymbol{e}^{(m)}$ have the same eigenvalues. In particular, it follows that the left and right Cauchy-Green strain tensors $\boldsymbol{B}$ and $\boldsymbol{C}$ have the same (positive) eigenvalues, and thus, by (2.51), the same first three principal invariants.

Note the following named measures of strain:

$$\boldsymbol{E} \equiv \boldsymbol{E}^{(2)} = \tfrac{1}{2}(\boldsymbol{U}^2 - \boldsymbol{I}) = \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}), \tag{2.86}$$

is the Green, or Lagrangian, or Green-Lagrange, or Green-Saint Venant strain tensor;

$$\boldsymbol{E}^{(1)} = \boldsymbol{U} - \boldsymbol{I}, \tag{2.87}$$

is the Biot, or engineering, or nominal strain tensor;

$$\boldsymbol{E}^{(0)} = \ln \boldsymbol{U}, \tag{2.88}$$

is the Hencky, or natural, or true strain tensor;

$$\boldsymbol{E}^{(-2)} = \tfrac{1}{2}(\boldsymbol{I} - \boldsymbol{U}^{-2}) = \tfrac{1}{2}(\boldsymbol{I} - \boldsymbol{C}^{-1}), \tag{2.89}$$

is the Almansi strain tensor;

$$\boldsymbol{e} \equiv \boldsymbol{e}^{(-2)} = \tfrac{1}{2}(\boldsymbol{I} - \boldsymbol{B}^{-1}) = \tfrac{1}{2}(\boldsymbol{I} - \boldsymbol{V}^{-2}). \tag{2.90}$$

is the Eulerian strain tensor; and

$$\boldsymbol{e}^{(0)} = \ln \boldsymbol{V}, \tag{2.91}$$

is the (Eulerian) logarithmic strain tensor.

Finally in this section it is useful to note that the *displacement* $\boldsymbol{u}$ of a particle is defined as

$$\boxed{\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X},} \tag{2.92}$$

so that

$$\boldsymbol{x} = \boldsymbol{X} + \boldsymbol{u} \tag{2.93}$$

and

$$\boldsymbol{F} = \operatorname{Grad} \boldsymbol{x} = \boldsymbol{I} + \operatorname{Grad} \boldsymbol{u}, \tag{2.94}$$

where Grad $\boldsymbol{u}$ is the *displacement gradient* (recall that Grad $\boldsymbol{X} = \boldsymbol{I}$, the identity tensor.) It follows that the Green tensor can be expressed in terms of the displacement as

$$\boldsymbol{E} = \tfrac{1}{2}\left[\operatorname{Grad}\boldsymbol{u} + (\operatorname{Grad}\boldsymbol{u})^T + (\operatorname{Grad}\boldsymbol{u})^T(\operatorname{Grad}\boldsymbol{u})\right], \tag{2.95}$$

or, in Cartesian components, as

$$E_{ij} = \tfrac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right]. \tag{2.96}$$

2.8 Homogeneous deformations

If $\boldsymbol{F}$ is independent of $\boldsymbol{X}$ then the deformation is said to be *homogeneous* (the same at each point of the body). The most general form of homogeneous deformation is given by integrating $d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$ as: $\boldsymbol{x} = \boldsymbol{F}\boldsymbol{X} + \boldsymbol{c}$, with $\boldsymbol{F}$ independent of $\boldsymbol{X}$ and $\boldsymbol{c}$ a constant vector.

Homogeneous deformations are *universal*: they can be performed for *any* elastic solid. They are also quite easy to implement experimentally. They thus form the basis for the standardized testing and evaluation of the elastic properties of solids. The following examples are all special and important cases of homogeneous deformations.

2.8.1 Simple elongation

Consider the uniform axial extension of a solid right circular cylinder (with lateral contraction). For this deformation there is no change in the orientation of the principal axes of $\boldsymbol{U}$ during the deformation and $\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V}$. Let the principal axis $\boldsymbol{u}^{(1)}$ lie along the cylinder axis and correspond to principal stretch λ_1. Then, since there is symmetry perpendicular to the axis, $\lambda_2 = \lambda_3$ and hence the deformation gradient has the following components in the $\boldsymbol{u}^{(i)}$ basis:

$$\boldsymbol{F} = \boldsymbol{U} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}. \tag{2.97}$$

The corresponding deformation is thus

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_2 X_3. \tag{2.98}$$

Hence a rod with material length L and radius A is deformed into a rod of spatial length l and radius a, with the connections

$$l = \lambda_1 L, \qquad a = \lambda_2 A. \tag{2.99}$$

In Table I we give an interpretation of several strain measures in terms of the changes in length of the rod.

Table I. Some measures of strain in simple elongation.

strain	definition	$\boldsymbol{u}^{(1)}\otimes\boldsymbol{u}^{(1)}$ component	or
Biot	$\boldsymbol{U}-\boldsymbol{I}$	$\lambda_1 - 1$	$\dfrac{l-L}{L}$
Green	$\frac{1}{2}(\boldsymbol{C}-\boldsymbol{I})$	$\frac{1}{2}(\lambda_1^2-1)$	$\dfrac{l^2-L^2}{2L^2}$
Almansi	$\frac{1}{2}(\boldsymbol{I}-\boldsymbol{B}^{-1})$	$\frac{1}{2}\left(1-\dfrac{1}{\lambda_1^2}\right)$	$\dfrac{l^2-L^2}{2l^2}$
Hencky	$\ln\boldsymbol{U}$	$\ln\lambda_1$	$\ln\left(\dfrac{l}{L}\right)$

Note that a for a small stretch, $\lambda_1 = 1+e$, where $|e| \ll 1$, and all the strains coincide as being equal to e.

For *incompressible* solids we must have $\det\boldsymbol{F} = 1$ at all times, so that $\lambda_1^2\lambda_2 = 1$, giving $\lambda_2 = \lambda_1^{-1/2}$. Hence an elongated ($\lambda_1 > 1$) incompressible rod contracts laterally ($\lambda_2 < 1$), and vice-versa.

2.8.2 Pure dilatation

This is defined by $\lambda_1 = \lambda_2 = \lambda_3$, $\boldsymbol{F} = \lambda_1\boldsymbol{I}$ and might be associated with, for example, the deformation of a cube into a cube of a different size or a sphere into another sphere.

Incompressibility gives $\lambda_1^3 = 1$, i.e. $\lambda_1 = 1$, and thus, incompressible materials cannot be subjected to pure dilatation.

2.8.3 Simple shear

Simple shear is defined by the equations

$$x_1 = X_1 + KX_2, \qquad x_2 = X_2, \qquad x_3 = X_3, \tag{2.100}$$

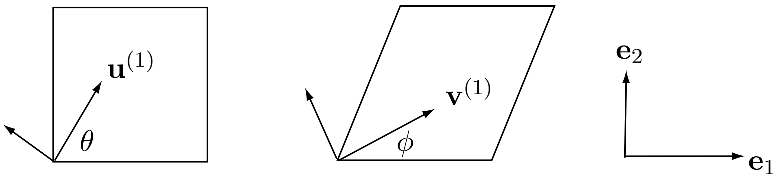

Figure 5: Simple shear in the (X_1, X_2) plane showing the orientation angles θ and ϕ of the Lagrangian and Eulerian principal axes.

where K (constant) is called the *amount of shear*, $\tan^{-1} K$ is the *angle of shear*, and the same basis vectors are used for both reference and current coordinates ($\boldsymbol{E}_i = \boldsymbol{e}_i$). See Figure 5.

The deformation gradient $\boldsymbol{F}$ has matrix of components, denoted F, given by

$$\mathsf{F} = \left(\frac{\partial x_i}{\partial X_j}\right) = \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.101}$$

To find the Lagrangian principal axes we consider, in matrix form,

$$\mathsf{U}^2 = \mathsf{F}^T\mathsf{F} = \begin{bmatrix} 1 & K & 0 \\ K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.102}$$

The characteristic equation for $\boldsymbol{U}^2$, from which the eigenvalues λ^2 are determined, is

$$\det(\boldsymbol{U}^2 - \lambda^2 \boldsymbol{I}) = 0, \tag{2.103}$$

i.e.

$$\begin{vmatrix} 1-\lambda^2 & K & 0 \\ K & 1+K^2-\lambda^2 & 0 \\ 0 & 0 & 1-\lambda^2 \end{vmatrix}, \tag{2.104}$$

or, when expanded out,

$$(\lambda^2 - 1)[\lambda^4 - (2 + K^2)\lambda^2 + 1] = 0. \tag{2.105}$$

Let the values of λ be denoted by λ_1, λ_2 and $\lambda_3 = 1$. Then

$$\lambda_1^2 + \lambda_2^2 = 2 + K^2, \qquad \lambda_1^2\lambda_2^2 = 1. \tag{2.106}$$

Now set $\lambda_1 \geq 1$, $\lambda_2 = \lambda_1^{-1}$ so that

$$\lambda_1^2 + \lambda_1^{-2} = 2 + K^2 \tag{2.107}$$

and hence

$$K = \lambda_1 - \lambda_1^{-1} = \lambda_2^{-1} - \lambda_2, \tag{2.108}$$

in which we have taken $K \geq 0$ to correspond to $\lambda_1 \geq 1$. It follows that the eigenvalues of $\boldsymbol{U}$ are

$$\boxed{\lambda_1 = \frac{K}{2} + \sqrt{1 + \frac{K^2}{4}}, \qquad \lambda_2 = -\frac{K}{2} + \sqrt{1 + \frac{K^2}{4}}, \qquad \lambda_3 = 1.} \tag{2.109}$$

Hence the greatest stretch is $\lambda_1 > 1$, the intermediate principal stretch is $\lambda_3 = 1$ and the least stretch is $\lambda_2 < 1$.

Now let us determine the directions where these principal stretches occur. Here we look for the Lagrangian (material) directions (i.e. the directions of the eigenvectors $\boldsymbol{u}^{(i)}$ of $\boldsymbol{U}$), but similar computations can be conducted for the Eulerian (spatial) principal directions (i.e. the directions of the eigenvectors $\boldsymbol{v}^{(i)}$ of $\boldsymbol{V}$).

We call θ the angle between $\boldsymbol{E}_1$ and $\boldsymbol{u}^{(1)}$, see Figure 5. Then

$$\boldsymbol{u}^{(1)} = \cos\theta\, \boldsymbol{E}_1 + \sin\theta\, \boldsymbol{E}_2, \qquad \boldsymbol{u}^{(2)} = -\sin\theta\, \boldsymbol{E}_1 + \cos\theta\, \boldsymbol{E}_2, \tag{2.110}$$

and conversely,

$$\boldsymbol{E}^{(1)} = \cos\theta\, \boldsymbol{u}^{(1)} - \sin\theta\, \boldsymbol{u}^{(2)}, \qquad \boldsymbol{E}^{(2)} = \sin\theta\, \boldsymbol{u}^{(1)} + \cos\theta\, \boldsymbol{u}^{(2)}. \tag{2.111}$$

Now we compute the quantities $\boldsymbol{E}_1 \cdot \boldsymbol{U}^2\, \boldsymbol{E}_1$, $\boldsymbol{E}_2 \cdot \boldsymbol{U}^2\, \boldsymbol{E}_2$, and $\boldsymbol{E}_1 \cdot \boldsymbol{U}^2\, \boldsymbol{E}_2$ in two different ways. First we use the components of $\boldsymbol{U}^2$ in ($\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$), see (2.102). We find that

$$\boldsymbol{E}_1 \cdot \boldsymbol{U}^2\, \boldsymbol{E}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & K & 0 \\ K & 1 + K^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ K \\ 0 \end{bmatrix} = 1, \tag{2.112}$$

and similarly, that

$$\boldsymbol{E}_2 \cdot \boldsymbol{U}^2\, \boldsymbol{E}_2 = 1 + K^2, \qquad \boldsymbol{E}_1 \cdot \boldsymbol{U}^2\, \boldsymbol{E}_2 = K. \tag{2.113}$$

Second we use the decomposition (2.111); hence

$$\boldsymbol{E}_1 \cdot \boldsymbol{U}^2 \boldsymbol{E}_1 = (\cos\theta\, \boldsymbol{u}^{(1)} - \sin\theta\, \boldsymbol{u}^{(2)}) \cdot (\lambda_1^2 \cos\theta\, \boldsymbol{u}^{(1)} - \lambda_2^2 \sin\theta\, \boldsymbol{u}^{(2)}) = \lambda_1^2 \cos^2\theta + \lambda_2^2 \sin^2\theta, \tag{2.114}$$

where we recalled that the $\boldsymbol{u}^{(i)}$ are orthonormal eigenvectors of $\boldsymbol{U}^2$ with associated eigenvalue λ_i^2. Similarly we find that

$$\boldsymbol{E}_2 \cdot \boldsymbol{U}^2\, \boldsymbol{E}_2 = \lambda_1^2 \sin^2\theta + \lambda_2^2 \cos^2\theta, \qquad \boldsymbol{E}_1 \cdot \boldsymbol{U}^2\, \boldsymbol{E}_2 = (\lambda_1^2 - \lambda_2^2) \cos\theta \sin\theta. \tag{2.115}$$

Comparison of the two sets of equations shows that

$$\begin{aligned} &\lambda_1^2 \cos^2\theta + \lambda_2^2 \sin^2\theta = 1, \\ &\lambda_1^2 \sin^2\theta + \lambda_2^2 \cos^2\theta = 1 + K^2, \\ &(\lambda_1^2 - \lambda_2^2) \sin\theta \cos\theta = K, \end{aligned} \tag{2.116}$$

from which we may deduce λ_1^2 and λ_2^2 (but we have already established them earlier) and also that (by subtracting the second equation from the first, and dividing the by the third)

$$\boxed{\tan 2\theta = -\frac{2}{K} \quad \left(\frac{\pi}{4} \leq \theta < \frac{\pi}{2}\right).} \tag{2.117}$$

The corresponding angle ϕ for the principal axes of $\boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{V}^2$ is calculated in a similar way. Let $\boldsymbol{v}^{(1)} = \cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2$, $\boldsymbol{v}^{(2)} = -\sin\phi\,\boldsymbol{e}_1 + \cos\phi\,\boldsymbol{e}_2$, see Figure 5. The end result is

$$\boxed{\tan 2\phi = \frac{2}{K} \quad \left(0 < \phi \leq \frac{\pi}{4}\right).} \tag{2.118}$$

Hence we have established the directions where the greatest and smallest stretches occur. For instance consider a block that is subject to simple shear of amount $K = 1$. Then its diagonal is at an angle $\alpha = \tan^{-1}(1/2) \simeq 26.6°$ from $\boldsymbol{e}_1$, and the greatest stretch occurs at an angle $\phi = \frac{1}{2}\tan^{-1}(2) \simeq 31.7°$ from $\boldsymbol{e}_1$, see Figure 6.

Figure 6: Block sheared by amount $K = 1$, with unit vectors $\boldsymbol{u}^{(1)}$, $\boldsymbol{u}^{(2)}$.

2.9 Analysis of motion

2.9.1 Velocity gradient tensor

Recalling that the velocity is denoted $\boldsymbol{v}$, we define the *velocity gradient tensor*, denoted $\boldsymbol{L}$, as

$$\boxed{\boldsymbol{L} = \operatorname{grad}\boldsymbol{v},} \tag{2.119}$$

which has components

$$L_{ij} = \frac{\partial v_i}{\partial x_j} \tag{2.120}$$

with respect to the basis $\{\boldsymbol{e}_i\}$.

Using the second identity in (2.31), we obtain

$$\mathrm{Grad}\,\boldsymbol{v} = (\mathrm{grad}\,\boldsymbol{v})\boldsymbol{F} = \boldsymbol{L}\boldsymbol{F}. \tag{2.121}$$

Since $\boldsymbol{v} = \dot{\boldsymbol{x}}$ we also have

$$\mathrm{Grad}\;\dot{\boldsymbol{x}} = \frac{\partial}{\partial t}\mathrm{Grad}\;\boldsymbol{x} = \dot{\boldsymbol{F}}, \tag{2.122}$$

recalling that the superposed dot represents the material time derivative. Hence, we have the important connection

$$\boxed{\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}.} \tag{2.123}$$

Using the result (proved below)

$$\frac{\partial}{\partial t}(\det \boldsymbol{F}) = (\det \boldsymbol{F})\,\mathrm{tr}\,(\boldsymbol{F}^{-1}\dot{\boldsymbol{F}}) \tag{2.124}$$

together with (2.123) we deduce that

$$\frac{\partial}{\partial t}(\det \boldsymbol{F}) = (\det \boldsymbol{F})\,\mathrm{tr}\,(\boldsymbol{L}) \tag{2.125}$$

or, equivalently,

$$\boxed{\dot{J} = J\,\mathrm{tr}\,(\boldsymbol{L}) = J\,\mathrm{div}\,\boldsymbol{v},} \tag{2.126}$$

remembering that $J = \det \boldsymbol{F}$, $\mathrm{tr}\,(\boldsymbol{L}) = L_{ii} = \partial v_i/\partial x_i = \mathrm{div}\,\boldsymbol{v}$.

Thus, $\mathrm{div}\,\boldsymbol{v}$ measures the rate at which volume changes during the motion. For an *isochoric* motion $J \equiv 1$, $\dot{J} = 0$ and hence

$$\mathrm{div}\,\boldsymbol{v} = 0. \tag{2.127}$$

In *incompressible* media all motions are isochoric and there, (2.127) must hold at all times.

It should also be noted that from (2.123) and the fact that $\boldsymbol{F}\boldsymbol{F}^{-1} = \boldsymbol{I}$ it follows that

$$\frac{\partial}{\partial t}(\boldsymbol{F}^{-1}) = -\boldsymbol{F}^{-1}\boldsymbol{L}. \tag{2.128}$$

Proof of (2.124) Let $\boldsymbol{F}$ be a second-order tensor depending on a scalar variable τ, and J be its determinant. For a small increment δ in τ we have

$$\boldsymbol{F}(\tau + \delta) = \boldsymbol{F}(\tau) + \delta\dot{\boldsymbol{F}}(\tau) + \mathcal{O}(\delta^2), \tag{2.129}$$

where the superposed dot denotes $\partial/\partial\tau$. Then

$$\dot{J} = \lim\nolimits_{\delta\to 0}\frac{1}{\delta}\left[J(\tau+\delta) - J(\tau)\right] = \lim\nolimits_{\delta\to 0}\frac{1}{\delta}\left[\det(\boldsymbol{F} + \delta\dot{\boldsymbol{F}}) - J\right]. \tag{2.130}$$

But

$$\begin{aligned} \det(\boldsymbol{F}+\delta\dot{\boldsymbol{F}}) &= \det[\boldsymbol{F}(\boldsymbol{I}+\delta\boldsymbol{F}^{-1}\dot{\boldsymbol{F}})] = (\det\boldsymbol{F})\det(\boldsymbol{I}+\delta\boldsymbol{F}^{-1}\dot{\boldsymbol{F}}) \\ &= J\det[\delta(\boldsymbol{F}^{-1}\dot{\boldsymbol{F}}+\delta^{-1}\boldsymbol{I})] = J\delta^3\det(\boldsymbol{F}^{-1}\dot{\boldsymbol{F}}+\delta^{-1}\boldsymbol{I}) \\ &= J\delta^3[-(-\delta^{-1})^3+I_1(-\delta^{-1})^2-I_2(-\delta^{-1})+I_3] \\ &= J[1+\delta\,I_1+\mathcal{O}(\delta^2)], \\ &= J[1+\delta\,\mathrm{tr}(\boldsymbol{F}^{-1}\dot{\boldsymbol{F}})+\mathcal{O}(\delta^2)], \end{aligned} \tag{2.131}$$

where the I_k are the first three principal invariants of $\boldsymbol{F}^{-1}\dot{\boldsymbol{F}}$, see (2.48) and (2.49). Now the limit (2.130) is easily computed as (2.124).

2.9.2 Stretching and spin

The deformation gradient $\boldsymbol{F}$ describes how material line elements change their length and orientation during deformation; the velocity gradient $\boldsymbol{L}$ describes the rate of these changes. Note that while $\boldsymbol{F}$ relates B_t to B_r, $\boldsymbol{L}$ is independent of B_r.

Let us write

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}, \tag{2.132}$$

where

$$\boldsymbol{D} = \underbrace{\tfrac{1}{2}(\boldsymbol{L}+\boldsymbol{L}^T)}_{\text{symmetric}}, \qquad \boldsymbol{W} = \underbrace{\tfrac{1}{2}(\boldsymbol{L}-\boldsymbol{L}^T)}_{\text{skewsymmetric}}. \tag{2.133}$$

In order to interpret $\boldsymbol{D}$ and $\boldsymbol{W}$, we consider the line element $d\boldsymbol{X} \to d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$. Then, we calculate

$$\begin{aligned} d\boldsymbol{x}\cdot d\boldsymbol{x} - d\boldsymbol{X}\cdot d\boldsymbol{X} &= (\boldsymbol{F}d\boldsymbol{X})\cdot(\boldsymbol{F}d\boldsymbol{X}) - d\boldsymbol{X}\cdot d\boldsymbol{X} \\ &= d\boldsymbol{X}\cdot(\boldsymbol{F}^T\boldsymbol{F}d\boldsymbol{X}) - d\boldsymbol{X}\cdot d\boldsymbol{X} \\ &= d\boldsymbol{X}\cdot(\boldsymbol{F}^T\boldsymbol{F}-\boldsymbol{I})d\boldsymbol{X}. \end{aligned}$$

From (2.123) it follows that

$$\begin{aligned} \frac{\partial}{\partial t}(d\boldsymbol{x}\cdot d\boldsymbol{x} - d\boldsymbol{X}\cdot d\boldsymbol{X}) &= d\boldsymbol{X}\cdot\frac{\partial}{\partial t}(\boldsymbol{F}^T\boldsymbol{F}-\boldsymbol{I})d\boldsymbol{X} = d\boldsymbol{X}\cdot(\boldsymbol{F}^T\dot{\boldsymbol{F}}+\dot{\boldsymbol{F}}^T\boldsymbol{F})d\boldsymbol{X} \\ &= d\boldsymbol{X}\cdot(\boldsymbol{F}^T\boldsymbol{L}\boldsymbol{F}+\boldsymbol{F}^T\boldsymbol{L}^T\boldsymbol{F})d\boldsymbol{X} \\ &= (\boldsymbol{F}d\boldsymbol{X})\cdot(\boldsymbol{L}+\boldsymbol{L}^T)\boldsymbol{F}d\boldsymbol{X} = 2d\boldsymbol{x}\cdot\boldsymbol{D}d\boldsymbol{x}. \end{aligned}$$

This shows that $\boldsymbol{D}$ measures the rate at which line elements are changing their squared lengths. It is called the (*Eulerian*) *strain-rate tensor* or *rate of stretching tensor*. The motion is *rigid* if and only if $\boldsymbol{D} = \boldsymbol{O}$.

Note: The tensor $\boldsymbol{W}$ is called the *body spin*, and can sometimes be interpreted as a measure of the *rate of rotation*.

2.10 Integration of tensors

We first summarize some results from vector calculus that will be needed subsequently. The *divergence theorem* is written

$$\boxed{\int_R \operatorname{div}\boldsymbol{u}\,dv = \int_{\partial R} \boldsymbol{u}\cdot\boldsymbol{n}\,da,} \tag{2.134}$$

where R is a domain in $\mathbb{R}^3$ and ∂R is its boundary (a closed surface), and $\boldsymbol{u}$ is a vector field. An alternative (and equivalent) form of the theorem is

$$\boxed{\int_R \operatorname{grad}\phi\,dv = \int_{\partial R} \phi\boldsymbol{n}\,da,} \tag{2.135}$$

where ϕ is a scalar field, or, in index notation,

$$\int_R \frac{\partial \phi}{\partial x_i} dv = \int_{\partial R} \phi n_i da. \tag{2.136}$$

In particular, (2.136) applies to the *components* (which are scalar fields) T_{pq} of any second-order Cartesian tensor. Thus,

$$\int_R \frac{\partial T_{pq}}{\partial x_i} dv = \int_{\partial R} T_{pq} n_i da. \tag{2.137}$$

Hence, (2.137) with $i = p$, gives

$$\int_R \frac{\partial T_{pq}}{\partial x_p} dv = \int_{\partial R} T_{pq} n_p da \tag{2.138}$$

or, in tensor notation,

$$\boxed{\int_R \operatorname{div}\boldsymbol{T}\,dv = \int_{\partial R} \boldsymbol{T}^T\boldsymbol{n}\,da.} \tag{2.139}$$

This is an *important formula* and will occur frequently in the remaining chapters of this volume.

In particular, it provides nice proofs for (2.32) and (2.33). Hence, we have

$$\begin{aligned} &\int_{R_r} \operatorname{Div}\boldsymbol{T}\,dV = \int_{\partial R_r} \boldsymbol{T}^T\boldsymbol{N}\,dA = \int_{\partial R_t} \boldsymbol{T}^T J^{-1}\boldsymbol{F}^T\boldsymbol{n}\,da \\ &= \int_{\partial R_t} \left(J^{-1}\boldsymbol{F}\boldsymbol{T}\right)^T \boldsymbol{n}\,da = \int_{R_t} \operatorname{div}\left(J^{-1}\boldsymbol{F}\boldsymbol{T}\right) dv = \int_{R_r} J\operatorname{div}\left(J^{-1}\boldsymbol{F}\boldsymbol{T}\right) dV, \end{aligned} \tag{2.140}$$

where we used (2.139) twice and Nanson's formula once.

2.11 Transport formulas

Let C_t, S_t and R_t denote curves, surfaces and regions in B_t, the current configuration of the body. Then, the following identities hold:

$$\frac{d}{dt}\int_{C_t}\phi d\boldsymbol{x} = \int_{C_t}(\dot{\phi}d\boldsymbol{x} + \phi\boldsymbol{L}d\boldsymbol{x}) = \int_{C_t}(\dot{\phi}\boldsymbol{I} + \phi\boldsymbol{L})d\boldsymbol{x}, \tag{2.141}$$

$$\frac{d}{dt}\int_{S_t}\phi\boldsymbol{n}da = \int_{S_t}\{[\dot{\phi} + \phi\mathrm{tr}\,(\boldsymbol{L})]\boldsymbol{n} - \phi\boldsymbol{L}^T\boldsymbol{n}\}da, \tag{2.142}$$

$$\frac{d}{dt}\int_{R_t}\phi dv = \int_{R_t}[\dot{\phi} + \phi\mathrm{tr}\,(\boldsymbol{L})]dv, \tag{2.143}$$

$$\frac{d}{dt}\int_{C_t}\boldsymbol{u}\cdot d\boldsymbol{x} = \int_{C_t}(\dot{\boldsymbol{u}} + \boldsymbol{L}^T\boldsymbol{u})\cdot d\boldsymbol{x}, \tag{2.144}$$

$$\frac{d}{dt}\int_{S_t}\boldsymbol{u}\cdot\boldsymbol{n}da = \int_{S_t}[\dot{\boldsymbol{u}} + \boldsymbol{u}\mathrm{tr}\,(\boldsymbol{L}) - \boldsymbol{L}\boldsymbol{u}]\cdot\boldsymbol{n}da, \tag{2.145}$$

$$\frac{d}{dt}\int_{R_t}\boldsymbol{u}dv = \int_{R_t}[\dot{\boldsymbol{u}} + \mathrm{tr}\,(\boldsymbol{L})\boldsymbol{u}]dv. \tag{2.146}$$

Proof Use the formulas $d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$, $\boldsymbol{n}da = J\boldsymbol{F}^{-T}\boldsymbol{N}dA$, $dv = JdV$, to convert the integrals over C_t, S_t, R_t in B_t to integrals over C_r, S_r, R_r in B_r, together with expressions for $\dot{\boldsymbol{F}}$ and $\dot{J}$. We illustrate the process by proving (2.145).

$$\begin{aligned}
\frac{d}{dt}\int_{S_t}\boldsymbol{u}\cdot\boldsymbol{n}da &= \frac{d}{dt}\int_{S_r}\boldsymbol{u}\cdot(J\boldsymbol{F}^{-T}\boldsymbol{N})dA \\
&\qquad\text{(note the integral is now over } S_r) \\
&= \frac{d}{dt}\int_{S_r}(J\boldsymbol{F}^{-1}\boldsymbol{u})\cdot\boldsymbol{N}\,dA \\
&\qquad\text{(using the definition of transpose)} \\
&= \int_{S_r}\underbrace{\frac{\partial}{\partial t}(J\boldsymbol{F}^{-1}\boldsymbol{u})}_{\text{at fixed } \boldsymbol{X}}\cdot\boldsymbol{N}dA \\
&= \int_{S_r}[J\boldsymbol{F}^{-1}\dot{\boldsymbol{u}} + \dot{J}\boldsymbol{F}^{-1}\boldsymbol{u} + J\partial(\boldsymbol{F}^{-1})/\partial t\,\boldsymbol{u}]\cdot\boldsymbol{N}\,dA.
\end{aligned}$$

From (2.126) we have $\dot{J} = J\mathrm{tr}\,(\boldsymbol{L})$, and from (2.128) we have $\partial(\boldsymbol{F}^{-1})/\partial t = -\boldsymbol{F}^{-1}\boldsymbol{L}$. Thus,

$$\begin{aligned}
\frac{d}{dt}\int_{S_t} \boldsymbol{u}\cdot\boldsymbol{n}da &= \int_{S_r} [J\boldsymbol{F}^{-1}\dot{\boldsymbol{u}} + J\mathrm{tr}\,(\boldsymbol{L})\boldsymbol{F}^{-1}\boldsymbol{u} - J\boldsymbol{F}^{-1}\boldsymbol{L}\boldsymbol{u}]\cdot\boldsymbol{N}\,dA \\
&= \int_{S_r} \{J\boldsymbol{F}^{-1}[\dot{\boldsymbol{u}} + \mathrm{tr}\,(\boldsymbol{L})\boldsymbol{u} - \boldsymbol{L}\boldsymbol{u}]\}\cdot\boldsymbol{N}\,dA \\
&= \int_{S_r} [\dot{\boldsymbol{u}} + \mathrm{tr}\,(\boldsymbol{L})\boldsymbol{u} - \boldsymbol{L}\boldsymbol{u}]\cdot(J\boldsymbol{F}^{-T}\boldsymbol{N})dA \\
&= \int_{S_t} [\dot{\boldsymbol{u}} + \mathrm{tr}\,(\boldsymbol{L})\boldsymbol{u} - \boldsymbol{L}\boldsymbol{u}]\cdot\boldsymbol{n}\,da \\
&\qquad \text{(converting back to an integral over } S_t).
\end{aligned}$$

The other formulas are established by following the same approach.

3 Balance laws and equations of motion

The mechanics of continuous media are described by equations which express the balance of mass, linear momentum, angular momentum and energy in a moving body. These balance equations apply to all bodies, solid or fluid, and each gives rise to field equations (differential equations for scalar, vector and tensor fields) for sufficiently smooth motions (or jump conditions across surfaces of discontinuity).

The fundamental concepts here are mass, force and energy.

3.1 Mass conservation

Let R_t be an arbitrary material region in the current configuration B_t. Its mass is

$$m = \int_{R_t} \rho dv, \tag{3.1}$$

where ρ is the (current) *mass density*. As R_t moves it always consists of the same material, so its mass does not change, i.e.

$$\frac{d}{dt}\int_{R_t} \rho dv = 0. \tag{3.2}$$

This is one form of the *conservation of mass equation*. From the transport formula (2.143) we obtain

$$\int_{R_t} (\dot{\rho} + \rho \operatorname{div} \boldsymbol{v}) dv = 0,$$

and, since R_t is arbitrary, it follows that

$$\boxed{\dot{\rho} + \rho \operatorname{div} \boldsymbol{v} = 0,} \tag{3.3}$$

at each point of the body (this deduction requires that the integrand is continuous). This is the local form of the mass conservation equation and it is also known as the *continuity equation*. Using the spatial description of the time derivative, we find the equivalent form $\partial\rho/\partial t + \boldsymbol{v}\cdot\mathrm{grad}\,\rho + \rho\mathrm{div}\,\boldsymbol{v} = 0$, or

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div} \rho v = 0.} \tag{3.4}$$

Recall, from (2.126), that $\dot{J} = J\mathrm{div}\,\boldsymbol{v}$. Substitution for $\mathrm{div}\,\boldsymbol{v}$ from (3.3) then gives $\rho\dot{J} + \dot{\rho}J = 0$, i.e. $\partial(\rho J)/\partial t = 0$. Thus, ρJ is constant for any material particle. In the reference configuration $J = 1$, so that $\rho J = \rho_r$, where ρ_r is the mass density in the reference configuration. Thus,

$$\boxed{\rho = J^{-1}\rho_r,} \tag{3.5}$$

which is yet another form of the mass conservation equation.

An alternative way to derive (3.5) is to note that the total mass is the same in the reference configuration and in the current configuration:

$$\int_{R_r} \rho_r dV = \int_{R_t} \rho dv = \int_{R_r} \rho J dV,$$

where R_r is the counterpart of R_t in the reference configuration.

3.2 Forces, moments and momentum

The concepts of *force* and *moment* describe the action on a moving material body $\mathcal{B}$ of its surroundings and the mutual actions of the parts of $\mathcal{B}$ on each other.

To a material region R_t in the current configuration B_t we associate two vectors, $\boldsymbol{F}(R_t)$ and $\boldsymbol{G}(R_t; o)$, called the total *force* and the total *moment with respect to o* on the material in R_t. Two types of forces and moments must be accounted for in general. These are *body forces* and *body moments*, which act on the particles of a body (arising from gravity or magnetic fields, for example), and *contact forces* and *contact torques* resulting from the action of one part of the body on another across a separating surface (for example, pressure or friction). The latter are also called *surface forces* and *surface moments*.

The *body* force and moment, measured *per unit mass*, are denoted $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively. Their contributions to $\boldsymbol{F}(R_t)$ and $\boldsymbol{G}(R_t; o)$ are

$$\int_{R_t} \rho \boldsymbol{b} dv, \qquad \int_{R_t} [\boldsymbol{x} \times (\rho \boldsymbol{b}) + \rho \boldsymbol{c}] dv \tag{3.6}$$

respectively, where $\boldsymbol{x}$ is the position vector of the point at which $\boldsymbol{b}$ acts.

A mathematical description of surface forces relies on *Cauchy's stress principle*, which is regarded as an axiom. This states that

> *the action of the material occupying that part of* B_t *exterior to a closed surface* S *on the material occupying the interior part is represented by a vector field, denoted* $\boldsymbol{t}_{(\boldsymbol{n})}$, *defined on* S *and with physical dimensions of force per unit area.*

This is depicted in Fig. 7. We refer to $\boldsymbol{t}_{(\boldsymbol{n})}$ as the *stress vector*. It is assumed to depend continuously on $\boldsymbol{n}$, the unit outward normal to S.

If this stress principle gives a complete account of surface action then the material is said to be *non-polar* and does not admit surface moments. All classical theories of solids and fluids are of this type.

The contributions to $\boldsymbol{F}(R_t)$ and $\boldsymbol{G}(R_t; o)$ of the surface forces acting on the boundary ∂R_t of R_t are

$$\int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} da, \qquad \int_{\partial R_t} \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{n})} da, \tag{3.7}$$

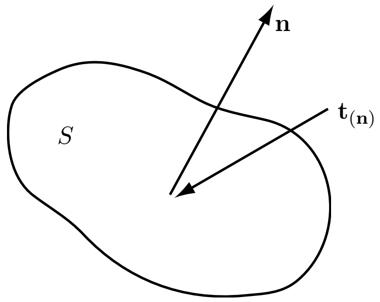

Figure 7: Stress vector $\boldsymbol{t}_{(\boldsymbol{n})}$ at a point of the surface S where the unit normal is $\boldsymbol{n}$.

respectively. We now have

$$\boldsymbol{F}(R_t) = \int_{R_t} \rho \boldsymbol{b} dv + \int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} da, \tag{3.8}$$

$$\boldsymbol{G}(R_t; o) = \int_{R_t} \rho(\boldsymbol{x} \times \boldsymbol{b} + \boldsymbol{c}) dv + \int_{\partial R_t} \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{n})} da \tag{3.9}$$

for the total force and total moment about o acting on R_t.

The *linear momentum* of the material occupying R_t is defined as

$$\boldsymbol{M}(R_t) = \int_{R_t} \rho \boldsymbol{v} dv. \tag{3.10}$$

With respect to an origin o, the *angular momentum* of R_t is defined as

$$\boldsymbol{H}(R_t; o) = \int_{R_t} \boldsymbol{x} \times (\rho \boldsymbol{v}) dv. \tag{3.11}$$

Now, *Euler's laws of motion* are

$$\frac{d\boldsymbol{M}}{dt} = \boldsymbol{F}, \qquad \frac{d\boldsymbol{H}}{dt} = \boldsymbol{G}, \tag{3.12}$$

and these hold independently of the choice of origin (although $\boldsymbol{G}$ and $\boldsymbol{H}$ do depend on such a choice). They parallel Newton's laws for particles and rigid bodies. Note, however, that in classical mechanics $(3.12)_2$ is a consequence of $(3.12)_1$, whereas in continuum mechanics this is not the case and the two equations in (3.12) are *independent*.

Here we do not consider body torques, so we set $\boldsymbol{c} = \boldsymbol{0}$, and equations (3.12) are then written in full as

$$\frac{d}{dt} \int_{R_t} \rho \boldsymbol{v} dv = \int_{R_t} \rho \boldsymbol{b} dv + \int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} da, \tag{3.13}$$

$$\frac{d}{dt} \int_{R_t} \rho \boldsymbol{x} \times \boldsymbol{v} dv = \int_{R_t} \rho \boldsymbol{x} \times \boldsymbol{b} dv + \int_{\partial R_t} \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{n})} da. \tag{3.14}$$

These are the equations of, respectively, *linear* and *angular momentum balance*.

Using transport formulas and (3.3), we obtain

$$\frac{d}{dt}\int_{R_t} \rho \boldsymbol{v} dv = \frac{d}{dt}\int_{R_r} \rho_r \boldsymbol{v} dV = \int_{R_r} \rho_r \dot{\boldsymbol{v}} dV = \int_{R_t} \rho \dot{\boldsymbol{v}} dv. \tag{3.15}$$

and

$$\frac{d}{dt}\int_{R_t} \rho(\boldsymbol{x} \times \boldsymbol{v}) dv = \frac{d}{dt}\int_{R_r} \rho_r(\boldsymbol{x} \times \boldsymbol{v}) dV$$
$$= \int_{R_r} \rho_r(\dot{\boldsymbol{x}} \times \boldsymbol{v} + \boldsymbol{x} \times \dot{\boldsymbol{v}}) dV = \int_{R_r} \rho_r(\boldsymbol{v} \times \boldsymbol{v} + \boldsymbol{x} \times \dot{\boldsymbol{v}}) dV = \int_{R_t} \rho(\boldsymbol{x} \times \dot{\boldsymbol{v}}) dv.$$

Hence (3.13) and (3.14) can be written

$$\int_{R_t} \rho(\boldsymbol{a} - \boldsymbol{b}) dv = \int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} da, \tag{3.16}$$

$$\int_{R_t} \rho \boldsymbol{x} \times (\boldsymbol{a} - \boldsymbol{b}) dv = \int_{\partial R_t} \boldsymbol{x} \times \boldsymbol{t}_{(\boldsymbol{n})} da, \tag{3.17}$$

where $\boldsymbol{a} \equiv \dot{\boldsymbol{v}}$ is the acceleration.

3.3 The theory of stress: Cauchy's theorem

Let $(\boldsymbol{t}_{(\boldsymbol{n})}, \boldsymbol{b})$ be a system of surface (contact) and body forces for $\mathcal{B}$ during a motion. A necessary and sufficient condition for the momentum balance equations (3.16) and (3.17) to be satisfied is that there exists a second-order tensor $\boldsymbol{\sigma}$, called the *Cauchy stress tensor*, such that

(i) for each unit vector $\boldsymbol{n}$,

$$\boxed{\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{\sigma}^T \boldsymbol{n},} \tag{3.18}$$

where $\boldsymbol{\sigma}$ is independent of $\boldsymbol{n}$,

(ii)

$$\boxed{\boldsymbol{\sigma}^T = \boldsymbol{\sigma},} \tag{3.19}$$

(iii) $\boldsymbol{\sigma}$ satisfies the *equation of motion*

$$\boxed{\operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \boldsymbol{a}.} \tag{3.20}$$

Proof Sufficiency: this can easily be checked by substituting (3.18)–(3.20) into (3.16) and (3.17). The calculations involved are similar to those required to prove necessity, so will not be given here. Necessity: assume that (3.16) and (3.17) are satisfied. The proof involves a number of steps. For this purpose, we now write $\boldsymbol{t}_{(\boldsymbol{n})}$ as $\boldsymbol{t}(\boldsymbol{n}, \boldsymbol{x})$ to indicate its dependence on both $\boldsymbol{n}$ and the position $\boldsymbol{x}$ on a surface.

Lemma Given any $\boldsymbol{x} \in \mathrm{B}_t$, any orthonormal basis $\{\boldsymbol{e}_i\}$ and any vector $\boldsymbol{p}$ with $\boldsymbol{p} \cdot \boldsymbol{e}_i > 0, i \in \{1, 2, 3\}$, the stress vector can be written

$$\boldsymbol{t}(\boldsymbol{p}, \boldsymbol{x}) = -\left[(\boldsymbol{p} \cdot \boldsymbol{e}_1)\boldsymbol{t}(-\boldsymbol{e}_1, \boldsymbol{x}) + (\boldsymbol{p} \cdot \boldsymbol{e}_2)\boldsymbol{t}(-\boldsymbol{e}_2, \boldsymbol{x}) + (\boldsymbol{p} \cdot \boldsymbol{e}_3)\boldsymbol{t}(-\boldsymbol{e}_3, \boldsymbol{x})\right].$$

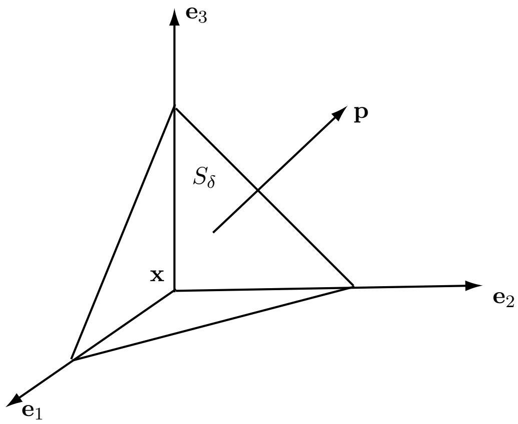

Figure 8: Tetrahedral volume bounded by the coordinate planes and the surface S_δ with unit normal $\boldsymbol{p}$. The point $\boldsymbol{x}$ is taken as the origin.

Proof of Lemma Suppose that $\boldsymbol{x} \in \mathrm{B}_t$, $\delta > 0$ and consider the tetrahedron shown in Fig. 8. The faces of the tetrahedron are denoted S_1, S_2, S_3 and S_δ, with unit (outward) normals $-\boldsymbol{e}_1$, $-\boldsymbol{e}_2$, $-\boldsymbol{e}_3$, $\boldsymbol{p}$ respectively, δ being the distance of the sloping face from $\boldsymbol{x}$. Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are continuous in B_t they are bounded on some neighbourhood of $\boldsymbol{x}$ in B_t containing the tetrahedron for sufficiently small δ. Similarly, ρ is bounded, so that

$$\left| \int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} da \right| = \left| \int_{R_t} \rho(\boldsymbol{a} - \boldsymbol{b}) dv \right| < k \times \mathrm{vol}\,(R_t),$$

where k is a constant independent of δ and $\mathrm{vol}\,(R_t)$ denotes the volume of the tetrahedron.

Let $A(\delta)$ denotes the area of the face S_δ. Then there exist positive constants c_1, c_2 such that

$$A(\delta) = c_1\delta^2, \quad \mathrm{vol}\,(R_t) = c_2\delta^3.$$

Hence

$$\frac{1}{A(\delta)} \int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} da \to 0 \quad \text{as } \delta \to 0.$$

Let A_i denote the area of S_i. Since, by the divergence theorem, we have

$$\int_{\partial R_t} \boldsymbol{e}_i \cdot \boldsymbol{n} dS = 0 \quad i \in \{1, 2, 3\},$$

it follows that

$$A_i = (\boldsymbol{e}_i \cdot \boldsymbol{p}) A(\delta).$$

But

$$\int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} da = \int_{S_\delta} \boldsymbol{t}(\boldsymbol{p}) da + \int_{S_1} \boldsymbol{t}(-\boldsymbol{e}_1) da + \int_{S_2} \boldsymbol{t}(-\boldsymbol{e}_2) da + \int_{S_3} \boldsymbol{t}(-\boldsymbol{e}_3) da$$

and

$$\frac{1}{A(\delta)}\int_{S_\delta} \boldsymbol{t}_{(\boldsymbol{n})} da \to \boldsymbol{t}(\boldsymbol{p},\boldsymbol{x}) \quad \text{as } \delta \to 0,$$
$$\frac{1}{A(\delta)}\int_{S_i} \boldsymbol{t}(-\boldsymbol{e}_i) da \to (\boldsymbol{e}_i \cdot \boldsymbol{p})\boldsymbol{t}(-\boldsymbol{e}_i,\boldsymbol{x}) \quad \text{as } \delta \to 0.$$

Hence the stated result.

It follows that

$$\boldsymbol{t}(\boldsymbol{e}_i,\boldsymbol{x}) = -\boldsymbol{t}(-\boldsymbol{e}_i,\boldsymbol{x})$$

and hence

$$\boldsymbol{t}(\boldsymbol{p},\boldsymbol{x}) = (\boldsymbol{p}\cdot\boldsymbol{e}_1)\boldsymbol{t}(\boldsymbol{e}_1,\boldsymbol{x}) + (\boldsymbol{p}\cdot\boldsymbol{e}_2)\boldsymbol{t}(\boldsymbol{e}_2,\boldsymbol{x}) + (\boldsymbol{p}\cdot\boldsymbol{e}_3)\boldsymbol{t}(\boldsymbol{e}_3,\boldsymbol{x}). \tag{3.21}$$

for *any* vector $\boldsymbol{p}$.

Proof of main result Consider the tensor $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma}^T(\boldsymbol{x}) = \boldsymbol{t}(\boldsymbol{e}_1,\boldsymbol{x})\otimes\boldsymbol{e}_1 + \boldsymbol{t}(\boldsymbol{e}_2,\boldsymbol{x})\otimes\boldsymbol{e}_2 + \boldsymbol{t}(\boldsymbol{e}_3,\boldsymbol{x})\otimes\boldsymbol{e}_3. \tag{3.22}$$

Then, by (3.21),

$$\boldsymbol{\sigma}^T\boldsymbol{n} = (\boldsymbol{e}_1\cdot\boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_1,\boldsymbol{x}) + (\boldsymbol{e}_2\cdot\boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_2,\boldsymbol{x}) + (\boldsymbol{e}_3\cdot\boldsymbol{n})\boldsymbol{t}(\boldsymbol{e}_3,\boldsymbol{x}) = \boldsymbol{t}(\boldsymbol{n},\boldsymbol{x}).$$

Hence (i) is established.

On substitution of $\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{\sigma}^T\boldsymbol{n}$ into (3.16), we obtain

$$\int_{R_t} \rho(\boldsymbol{a}-\boldsymbol{b})dv = \int_{\partial R_t} \boldsymbol{\sigma}^T\boldsymbol{n}da = \int_{R_t} \operatorname{div}\boldsymbol{\sigma}dv$$

by the divergence theorem. Thus,

$$\int_{R_t} [\operatorname{div}\boldsymbol{\sigma} - \rho(\boldsymbol{a}-\boldsymbol{b})]dv = \boldsymbol{0}.$$

Since R_t is arbitrary, (iii) follows (provided the above integrand is continuous). It remains to prove (ii).

Next, substitute (3.18) and (3.20) into (3.17) to give

$$\int_{R_t} \boldsymbol{x}\times(\operatorname{div}\boldsymbol{\sigma})dv = \int_{\partial R_t} \boldsymbol{x}\times(\boldsymbol{\sigma}^T\boldsymbol{n})da. \tag{3.23}$$

Noting that $\boldsymbol{u}\times\boldsymbol{v} = \boldsymbol{a}\times\boldsymbol{b}$, for any vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{a},\boldsymbol{b}$, is equivalent to

$$\boldsymbol{u}\otimes\boldsymbol{v} - \boldsymbol{v}\otimes\boldsymbol{u} = \boldsymbol{a}\otimes\boldsymbol{b} - \boldsymbol{b}\otimes\boldsymbol{a},$$

we write (3.23) as

$$\int_{R_t} [\boldsymbol{x}\otimes\operatorname{div}\boldsymbol{\sigma} - (\operatorname{div}\boldsymbol{\sigma})\otimes\boldsymbol{x}]dv = \int_{\partial R_t} (\boldsymbol{x}\otimes\boldsymbol{\sigma}^T\boldsymbol{n} - \boldsymbol{\sigma}^T\boldsymbol{n}\otimes\boldsymbol{x})da,$$

which, by application of the divergence theorem, becomes

$$\int_{R_t} [(\operatorname{grad} \boldsymbol{x})\boldsymbol{\sigma} - \boldsymbol{\sigma}^T (\operatorname{grad} \boldsymbol{x})^T] dv.$$

Since $\operatorname{grad} \boldsymbol{x} = \boldsymbol{I}$, the identity tensor, we deduce that

$$\int_{R_t} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^T) dv = \boldsymbol{O}.$$

Since R_t is arbitrary, (ii) follows.

3.4 Normal and shear stresses

Suppose an element of area da on a surface S with unit normal $\boldsymbol{n}$ is subjected to a contact force $\boldsymbol{t}_{(\boldsymbol{n})} da$. The *normal component* of the stress vector, denoted σ, is defined as

$$\boxed{\sigma \equiv \boldsymbol{n} \cdot \boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{n} \cdot (\boldsymbol{\sigma} \boldsymbol{n}).} \tag{3.24}$$

This is called the *normal stress* on the surface S. It is tensile when positive and compressive when negative.

The stress vector tangential to S is denoted $\boldsymbol{\tau}$, with magnitude τ, and given by

$$\boxed{\boldsymbol{\tau} \equiv \boldsymbol{t}_{(\boldsymbol{n})} - \sigma \boldsymbol{n}, \qquad \tau = |\boldsymbol{t}_{(\boldsymbol{n})} - \sigma \boldsymbol{n}|,} \tag{3.25}$$

from which it follows that

$$\tau^2 = \boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{t}_{(\boldsymbol{n})} - [\boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{n}]^2.$$

We refer to τ as the *shear stress*.

At a point $\boldsymbol{x}$ in the current configuration B_t let $\boldsymbol{\sigma}$ have components σ_{ij} with respect to basis vectors $\{\boldsymbol{e}_i\}$. Then σ_{ij} is the jth component of force per unit area in B_t acting on a surface whose normal is in the i-direction. In particular, for a surface normal to $\boldsymbol{e}_1$, σ_{11} is normal and σ_{12} and σ_{13} are tangential, i.e. they are shearing components. Similarly for the other components.

3.5 Principal stresses and principal axes of stress

Similarly to strain, we can find the principal stresses and principal axes of stress by solving the eigenproblem

$$\boxed{\boldsymbol{\sigma} \boldsymbol{n} = \sigma \boldsymbol{n},} \tag{3.26}$$

where the σ's are scalars and the $\boldsymbol{n}$'s are unit vectors. In practice, we solve first $\det(\boldsymbol{\sigma} - \sigma \boldsymbol{I}) = 0$ and find the principal stresses σ_1, σ_2, σ_3. Then we normalize the corresponding eigenvectors to be of unit length. Because $\boldsymbol{\sigma}$ is symmetric, they are orthogonal and we take them to form a direct orthonormal basis. In contrast to the eigenvalues of the strain tensors, $\boldsymbol{\sigma}$ is not necessarily positive definite and the σ_i may be of any sign.

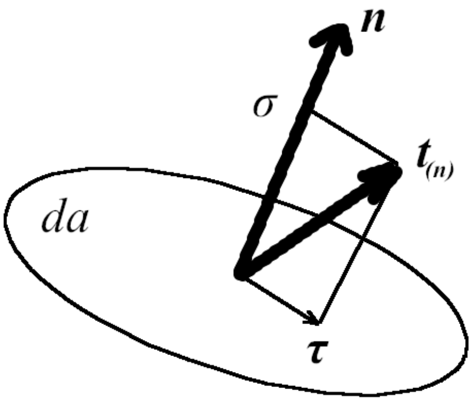

Figure 9: The stress vector can be decomposed into the normal stress and the shear stress.

By the spectral theorem, the stress tensor has a diagonal representation in the coordinate system of its eigenvectors:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \tag{3.27}$$

in the $(\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3)$ basis.

3.6 Some states of stress

If $\tau = 0$ and σ is independent of $\boldsymbol{n}$ then the stress is said to be *hydrostatic* or *isotropic*. In this case there is a scalar field p, called the *hydrostatic pressure*, such that

$$\boxed{\boldsymbol{t}_{(\boldsymbol{n})} = -p\boldsymbol{n}, \qquad \boldsymbol{\sigma} = -p\boldsymbol{I}.} \tag{3.28}$$

In other words, the stress for a media subject to hydrostatic pressure has representation

$$\boldsymbol{\sigma} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}, \tag{3.29}$$

in any rectangular coordinate system. Here, $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$ and $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$, so the only component of force acting is a normal pressure; there is no shearing component.

Recall that the equations of equilibrium read: $\operatorname{div}\boldsymbol{\sigma} + \rho\boldsymbol{b} = \boldsymbol{0}$. In the absence of body forces they read: $\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{0}$ and these are automatically satisfied when $\boldsymbol{\sigma}$ has constant components, i.e. when the stress is *uniform*. Here we present three of the most common states of uniform stress, which form the basis of an experimental determination of the stress-strain relationships.

When a body is in a state of *uniform tension or compression*, it is described by

$$\boxed{\boldsymbol{\sigma} = T\boldsymbol{n} \otimes \boldsymbol{n},} \tag{3.30}$$

where $\boldsymbol{n}$ is a unit vector and T is a scalar: $T > 0$ corresponds to a uniaxial tension in the direction of $\boldsymbol{n}$, and $T < 0$ corresponds to a compression. Take two unit vectors $\boldsymbol{p}$, $\boldsymbol{q}$ such that $(\boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q})$ forms a direct orthonormal basis. Then the components of σ in that basis are

$$\boldsymbol{\sigma} = \begin{bmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.31}$$

When a body is in a state of *uniform shear stress*, in the $\boldsymbol{p}$ direction, on the planes $\boldsymbol{q} \cdot \boldsymbol{x} = 0$, it is described by

$$\boxed{\boldsymbol{\sigma} = S(\boldsymbol{p} \otimes \boldsymbol{q} + \boldsymbol{q} \otimes \boldsymbol{p}),} \tag{3.32}$$

where S is a constant scalar. In the direct orthonormal basis $(\boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q})$, where $\boldsymbol{n} = \boldsymbol{p} \times \boldsymbol{q}$, the components of σ are

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & S & 0 \\ S & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.33}$$

Finally, consider a thin sheet of material, and apply uniform forces on its edges, while keeping its faces free of traction. Then it is reasonable to assume that the stress is of the form

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{12} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.34}$$

where the non-zero components σ_{11}, σ_{12}, σ_{22} are constants. The membrane is then said to be in a state of *uniform plane stress.*

3.7 Stress tensors

Using Nanson's formula (2.47) the traction on an area element $\boldsymbol{n}da$ in the current configuration can be written

$$\boldsymbol{t}_{(\boldsymbol{n})}da = \boldsymbol{\sigma}^T\boldsymbol{n}da = J\boldsymbol{\sigma}^T\boldsymbol{F}^{-T}\boldsymbol{N}dA \equiv \boldsymbol{S}^T\boldsymbol{N}dA, \tag{3.35}$$

wherein the *nominal stress tensor* $\boldsymbol{S}$ is defined as

$$\boxed{\boldsymbol{S} = J\boldsymbol{F}^{-1}\boldsymbol{\sigma}.} \tag{3.36}$$

This is also referred to as the *engineering stress*, while the following stress tensor $\boldsymbol{P} \equiv \boldsymbol{S}^T$, or

$$\boxed{\boldsymbol{P} = J\boldsymbol{\sigma}(\boldsymbol{F}^{-1})^T,} \tag{3.37}$$

is the so-called *first Piola-Kirchhoff stress tensor.* The nominal stress measures the force *per unit reference area* while $\boldsymbol{\sigma}$ measures the force *per unit deformed area.*

During a tensile test, a specimen is clamped and stretched. In general the stretch is measured in real-time throughout the deformation, and so is the force, by using a load-cell. To compute the Cauchy stress, one would need to measure the cross-section area of the specimen in real-time as well, which is no easy task. In contrast, the nominal stress tensor is easy to compute, simply by dividing the measured force by the original cross-section area.

The equation of motion

$$\operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \boldsymbol{a} = \rho \dot{\boldsymbol{v}},$$

can be recast in terms of the nominal stress $\boldsymbol{S}$. One way to do this is to use the integral form of the balance equation, i.e.

$$\int_{R_t} \rho \boldsymbol{b} dv + \int_{\partial R_t} \boldsymbol{\sigma} \boldsymbol{n} da = \int_{R_t} \rho \dot{\boldsymbol{v}} dv,$$

and convert the integrals over the current configuration to integrals over the reference configuration using mass conservation in the form $\rho dv = \rho_r dV$ and Nanson's formula (2.47). This leads to

$$\int_{R_r} \rho_r \boldsymbol{b} dV + \int_{\partial R_r} \boldsymbol{S}^T \boldsymbol{N} dA = \int_{R_r} \rho_r \dot{\boldsymbol{v}} dV,$$

and hence, by the divergence theorem,

$$\boxed{\operatorname{Div} \boldsymbol{S} + \rho_r \boldsymbol{b} = \rho_r \dot{\boldsymbol{v}}.} \tag{3.38}$$

Note that, in general, $\boldsymbol{S}$ is not symmetric but satisfies the connection

$$\boldsymbol{F} \boldsymbol{S} = \boldsymbol{S}^T \boldsymbol{F}^T \tag{3.39}$$

arising from symmetry of $\boldsymbol{\sigma}$. Similarly, the first Piola-Kirchhoff stress tensor (3.37) is not symmetric, which motivates the construction of the following stress tensor, $\boldsymbol{T}^{(2)} \equiv \boldsymbol{F}^{-1} \boldsymbol{P}$, or

$$\boxed{\boldsymbol{T}^{(2)} = J \boldsymbol{F}^{-1} \boldsymbol{\sigma} (\boldsymbol{F}^{-1})^T,} \tag{3.40}$$

the so-called *second Piola-Kirchhoff stress tensor.*

3.8 The energy equation

The *kinetic energy* $K(R_t)$ of the material occupying R_t is defined as

$$K(R_t) = \int_{R_t} \tfrac{1}{2} \rho \boldsymbol{v} \cdot \boldsymbol{v} \, dv, \tag{3.41}$$

and the *rate of working*, or *power*, $P(R_t)$ of the forces acting on R_t is defined as

$$P(R_t) = \int_{R_t} \rho \boldsymbol{b} \cdot \boldsymbol{v} \, dv + \int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{v} \, da. \tag{3.42}$$

By using $\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{\sigma n}$, we obtain

$$\begin{aligned}
P(R_t) &= \int_{R_t} \rho \boldsymbol{b} \cdot \boldsymbol{v} \, dv + \int_{\partial R_t} (\boldsymbol{\sigma n}) \cdot \boldsymbol{v} \, da \\
&= \int_{R_t} \rho \boldsymbol{b} \cdot \boldsymbol{v} \, dv + \int_{\partial R_t} (\boldsymbol{\sigma v}) \cdot \boldsymbol{n} \, da \\
&\qquad \text{(since } \boldsymbol{\sigma} \text{ is symmetric)} \\
&= \int_{R_t} [\rho \boldsymbol{b} \cdot \boldsymbol{v} + \operatorname{div}(\boldsymbol{\sigma v})] dv \\
&\qquad \text{(by the divergence theorem)} \\
&= \int_{R_t} [\rho \boldsymbol{b} \cdot \boldsymbol{v} + (\operatorname{div} \boldsymbol{\sigma}) \cdot \boldsymbol{v} + \operatorname{tr}(\boldsymbol{\sigma L})] dv \\
&\qquad \text{(since } (\sigma_{ij} v_j)_{,i} = \sigma_{ij,i} v_j + \sigma_{ij} v_{j,i} = \sigma_{ij,i} v_j + \sigma_{ij} L_{ji}) \\
&= \int_{R_t} [(\rho \boldsymbol{b} + \operatorname{div} \boldsymbol{\sigma}) \cdot \boldsymbol{v} + \operatorname{tr}(\boldsymbol{\sigma D})] dv \\
&\qquad \text{(since } \sigma_{ij} L_{ji} = \sigma_{ij}(D_{ij} + W_{ij}) = \sigma_{ij} D_{ij}) \\
&= \int_{R_t} [\rho \dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \operatorname{tr}(\boldsymbol{\sigma D})] dv \\
&\qquad \text{(using the equation of motion)} \\
&= \int_{R_t} \tfrac{1}{2} \rho \partial(\boldsymbol{v} \cdot \boldsymbol{v})/\partial t \, dv + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma D}) dv \\
&= \int_{R_r} \tfrac{1}{2} \rho_r \partial(\boldsymbol{v} \cdot \boldsymbol{v})/\partial t \, dV + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma D}) dv \\
&\qquad \text{(since } \rho dv = \rho_r dV) \\
&= \frac{d}{dt} \int_{R_r} \tfrac{1}{2} \rho_r (\boldsymbol{v} \cdot \boldsymbol{v}) dV + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma D}) dv \\
&= \frac{d}{dt} K(R_t) + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma D}) dv.
\end{aligned} \tag{3.43}$$

Thus,

$$\boxed{\int_{R_t} \rho \boldsymbol{b} \cdot \boldsymbol{v} dv + \int_{\partial R_t} \boldsymbol{t}_{(\boldsymbol{n})} \cdot \boldsymbol{v} \, da = \frac{d}{dt} \int_{R_r} \tfrac{1}{2} \rho_r (\boldsymbol{v} \cdot \boldsymbol{v}) dV + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma D}) dv.} \tag{3.44}$$

or

$$P(R_t) = \frac{d}{dt} K(R_t) + S(R_t), \tag{3.45}$$

where

$$S(R_t) = \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma D}) dv. \tag{3.46}$$

Equation (3.45) is an *energy balance equation*. The work done by the body and surface forces is converted into kinetic energy and $S(R_t)$. The latter may consist of stored (or potential) energy or be a measure of the amount of work dissipated in the form of heat or be a mixture of the two.

3.9 Conjugate measures of strain and stress

The expression (3.46) occurring in the energy balance equations can be converted to an integral over the reference configuration, as

$$\int_{R_r} J \,\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{D})dV. \tag{3.47}$$

The integrand in (3.47) is the rate of working of the stresses per unit reference volume (i.e. the stress power density). Using the symmetry of $\boldsymbol{\sigma}$, we have

$$J \,\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{D}) = J \,\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{L}) = \mathrm{tr}\,(\boldsymbol{F}\boldsymbol{S}\boldsymbol{L}) = \mathrm{tr}\,(\boldsymbol{S}\boldsymbol{L}\boldsymbol{F}) = \mathrm{tr}\,(\boldsymbol{S}\dot{\boldsymbol{F}}). \tag{3.48}$$

This shows that the stress power is also given by $\mathrm{tr}\,(\boldsymbol{S}\dot{\boldsymbol{F}})$. Because of this connection $\boldsymbol{S}$ and $\boldsymbol{F}$ are said to constitute a pair of *conjugate stress and deformation tensors.*

Furthermore, by recalling the definition (2.86) and writing

$$\boldsymbol{E}^{(2)} = \tfrac{1}{2}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}), \tag{3.49}$$

we obtain

$$\dot{\boldsymbol{E}}^{(2)} = \tfrac{1}{2}(\boldsymbol{F}^T\dot{\boldsymbol{F}} + \dot{\boldsymbol{F}}^T\boldsymbol{F}) \equiv \boldsymbol{F}^T\boldsymbol{D}\boldsymbol{F}. \tag{3.50}$$

This is used to write the stress power as

$$\mathrm{tr}\,(\boldsymbol{S}\dot{\boldsymbol{F}}) = \mathrm{tr}\left[\boldsymbol{S}(\boldsymbol{F}^T)^{-1}\boldsymbol{F}^T\dot{\boldsymbol{F}}\right] = \mathrm{tr}\left[\boldsymbol{S}(\boldsymbol{F}^T)^{-1}\dot{\boldsymbol{E}}^{(2)}\right] = \mathrm{tr}\left[\boldsymbol{T}^{(2)}\dot{\boldsymbol{E}}^{(2)}\right] \tag{3.51}$$

using the symmetry of $\boldsymbol{S}(\boldsymbol{F}^T)^{-1}$. Here we see the appearance of the *second Piola-Kirchhoff stress tensor* $\boldsymbol{T}^{(2)}$, defined through (3.40). The stress and strain pair $(\boldsymbol{T}^{(2)}, \boldsymbol{E}^{(2)})$ is a pair of *conjugate stress and strain tensors.*

Similarly, from $\boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{U}^2$ we also have

$$\dot{\boldsymbol{E}}^{(2)} = \tfrac{1}{2}(\boldsymbol{U}\dot{\boldsymbol{U}} + \dot{\boldsymbol{U}}\boldsymbol{U}),$$

and hence, using the symmetry of $\boldsymbol{T}^{(2)}$ and of $\dot{\boldsymbol{U}}$,

$$\mathrm{tr}\left[\boldsymbol{T}^{(2)}\dot{\boldsymbol{E}}^{(2)}\right] = \mathrm{tr}\left[\boldsymbol{T}^{(2)}\boldsymbol{U}\dot{\boldsymbol{U}}\right] = \mathrm{tr}\left[\tfrac{1}{2}(\boldsymbol{T}^{(2)}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{T}^{(2)})\dot{\boldsymbol{U}}\right].$$

This motivates the definition of the *Biot stress tensor* $\boldsymbol{T}^{(1)}$, conjugate to the strain tensor $\boldsymbol{E}^{(1)} \equiv \boldsymbol{U} - \boldsymbol{I}$, as

$$\boldsymbol{T}^{(1)} = \tfrac{1}{2}(\boldsymbol{T}^{(2)}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{T}^{(2)}), \tag{3.52}$$

which, by using the polar decomposition (2.64), may also be written as

$$\boxed{\boldsymbol{T}^{(1)} = \tfrac{1}{2}(\boldsymbol{S}\boldsymbol{R} + \boldsymbol{R}^T\boldsymbol{S}^T).} \tag{3.53}$$

We now have the connections

$$J \,\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{D}) = \mathrm{tr}\left[\boldsymbol{S}\dot{\boldsymbol{F}}\right] = \mathrm{tr}\left[\boldsymbol{T}^{(2)}\dot{\boldsymbol{E}}^{(2)}\right] = \mathrm{tr}\left[\boldsymbol{T}^{(1)}\dot{\boldsymbol{E}}^{(1)}\right]. \tag{3.54}$$

More generally, the (symmetric) stress tensor $\boldsymbol{T}^{(m)}$ conjugate to the strain tensor $\boldsymbol{E}^{(m)} \equiv (\boldsymbol{U}^m - \boldsymbol{I})/m$ discussed in Section 2.7 may be defined via the identity

$$\mathrm{tr}\left[\boldsymbol{T}^{(m)}\dot{\boldsymbol{E}}^{(m)}\right] = \mathrm{tr}\left[\boldsymbol{T}^{(1)}\dot{\boldsymbol{E}}^{(1)}\right] = \mathrm{tr}\left[\boldsymbol{T}^{(1)}\dot{\boldsymbol{U}}\right], \tag{3.55}$$

and it should be noted that this definition is independent of any material constitutive law.

4 Constitutive equations

4.1 Introduction

So far, we have established the following equations governing the motion of a continuous body:

- *equation of mass conservation*

$$\boxed{\dot{\rho} + \rho \operatorname{div} \boldsymbol{v} = 0;} \tag{4.1}$$

- *equation of motion*

$$\boxed{\operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \dot{\boldsymbol{v}};} \tag{4.2}$$

- *equation of angular momentum balance*

$$\boxed{\boldsymbol{\sigma}^T = \boldsymbol{\sigma}.} \tag{4.3}$$

These provide 7 scalar equations for 13 scalar fields — ρ, $\boldsymbol{v}$ (3 components), and $\boldsymbol{\sigma}$ (9 components), with the body force $\boldsymbol{b}$ regarded as known. Equivalently, given (4.3), equations (4.1) and (4.2) provide 4 equations for 10 scalar fields — ρ, $\boldsymbol{v}$ (3 components), and $\boldsymbol{\sigma}$ (6 components).

The missing 6 equations are provided in the form of *constitutive equations*, which give expressions for the 6 components of $\boldsymbol{\sigma}$ in terms of kinematical quantities, as we now describe.

It is assumed that at time t the stress is uniquely determined by the motion $\boldsymbol{\chi}$, i.e. $\boldsymbol{\sigma}$ is a function of $\boldsymbol{x}$, $\boldsymbol{v}$, $\boldsymbol{F}$, $\boldsymbol{L}, \dots$, and possibly also higher gradients of the deformation. We then have 10 equations for 10 unknowns, and, by substituting the constitutive equations into (4.1) and (4.2) we arrive at 4 equations for ρ and $\boldsymbol{v}$, and (4.3) will be satisfied automatically. We now illustrate the general principles involved in the development of constitutive equations by focussing on the case of homogeneous elastic materials, for which $\boldsymbol{\sigma}$ depends only on $\boldsymbol{F}$ (For an inhomogeneous material there is, additionally, explicit dependence on $\boldsymbol{X}$.)

Solids for which $\boldsymbol{\sigma}$ depends on $\boldsymbol{F}$ only are said to be *Cauchy elastic*. In what follows we focus on *Green elasticity*, or *hyperelasticity*, for which the stress is derived from a stored energy function W, say, which depends on $\boldsymbol{F}$ only: $W = W(\boldsymbol{F})$.

4.2 Hyperelastic materials

Recall, from Section 3.8, that the energy balance equation can be written in the form

$$\int_{R_t} \rho \boldsymbol{b} \cdot \boldsymbol{v} dv + \int_{\partial R_t} \boldsymbol{t} \cdot \boldsymbol{v} da = \frac{d}{dt} \int_{R_t} \tfrac{1}{2} \rho \boldsymbol{v} \cdot \boldsymbol{v} dv + \int_{R_t} \operatorname{tr}(\boldsymbol{\sigma} \boldsymbol{L}) dv. \tag{4.4}$$

If there is no dissipation then the work done by the body and surface forces is converted into kinetic energy and stored elastic energy. In this connection an interpretation for the second term on the right-hand side of (4.4) is needed.

Write

$$\int_{R_t} \mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{L})dv = \int_{R_r} J\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{L})dV. \tag{4.5}$$

Then, the integrand $J\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{L})$ is interpreted as the rate of increase of elastic energy per unit volume in B_r.

This prompts the introduction of the *elastic stored energy* $W(\boldsymbol{F})$ per unit volume in the reference configuration B_r such that

$$\frac{\partial}{\partial t}W(\boldsymbol{F}) = J\,\mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{L}) = \mathrm{tr}\,(\boldsymbol{S}\dot{\boldsymbol{F}}), \tag{4.6}$$

where the last equality follows from (3.48). Note that $W(\boldsymbol{F})$ is also referred to as the *strain energy* or *potential energy* (per unit volume in B_r). Then, we have

$$\int_{R_t} \mathrm{tr}\,(\boldsymbol{\sigma}\boldsymbol{L})dv = \int_{R_r} \frac{\partial}{\partial t}W(\boldsymbol{F})dV = \frac{d}{dt}\int_{R_r} W(\boldsymbol{F})dV = \frac{d}{dt}\int_{R_t} J^{-1}W(\boldsymbol{F})dv, \tag{4.7}$$

and

$$\int_{R_r} W(\boldsymbol{F})dV = \int_{R_t} J^{-1}W(\boldsymbol{F})dv \tag{4.8}$$

is the total elastic strain energy in the region R_r. The right-hand side of (4.4) can now be written as

$$\frac{d}{dt}(\text{kinetic energy} + \text{strain energy}). \tag{4.9}$$

Since W depends only on $\boldsymbol{F}$, we have

$$\frac{\partial}{\partial t}W(\boldsymbol{F}) = \frac{\partial W}{\partial F_{ij}}\frac{\partial F_{ij}}{\partial t} \equiv \mathrm{tr}\left(\frac{\partial W}{\partial \boldsymbol{F}}\dot{\boldsymbol{F}}\right), \tag{4.10}$$

where $\partial W/\partial \boldsymbol{F}$ is the second-order tensor with components defined by the convention

$$\left(\frac{\partial W}{\partial \boldsymbol{F}}\right)_{ji} = \frac{\partial W}{\partial F_{ij}}. \tag{4.11}$$

The expressions (4.6) and (4.10) must coincide for all $\dot{\boldsymbol{F}}$, leading to the conclusion that

$$\boxed{\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{F}}, \qquad S_{ji} = \frac{\partial W}{\partial F_{ij}},} \tag{4.12}$$

which is a quite simple formula for the nominal stress.

By recalling the connection (3.36) between the Cauchy stress $\boldsymbol{\sigma}$ and the nominal stress $\boldsymbol{S}$, we obtain

$$\boxed{\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\frac{\partial W}{\partial \boldsymbol{F}}, \qquad \sigma_{ij} = J^{-1}F_{ik}\frac{\partial W}{\partial F_{jk}},} \tag{4.13}$$

which provides a formula for $\boldsymbol{\sigma}$ in terms of $W(\boldsymbol{F})$.

We remark that $W(\boldsymbol{F})$ represents the work done (per unit volume at $\boldsymbol{X}$) by the stress in deforming the material from B_r to B_t (i.e. from $\boldsymbol{I}$ to $\boldsymbol{F}$) and is independent of the path taken in deformation space.

4.3 Objectivity

Suppose that a rigid-body motion

$$\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t) \tag{4.14}$$

is superimposed on the motion $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$. How do physical quantities change under this superposed motion? Let us look at some examples.

- Distances: According to (2.4), we have

$$\boxed{|\boldsymbol{x}^* - \boldsymbol{y}^*| = |\boldsymbol{x} - \boldsymbol{y}|,} \tag{4.15}$$

  so that distances are unchanged.

- Deformation gradient: The deformation gradient, $\boldsymbol{F}^* = \partial\boldsymbol{x}^*/\partial\boldsymbol{X}$, corresponding to the complete motion, is given by

$$\boxed{\boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F}.} \tag{4.16}$$

  In index notation, this may be proved as follows. Note first that, since

$$x_i^* = Q_{ip}x_p + c_i, \tag{4.17}$$

  we obtain

$$\frac{\partial x_i^*}{\partial x_k} = Q_{ip}\frac{\partial x_p}{\partial x_k} = Q_{ip}\delta_{pk} = Q_{ik}, \tag{4.18}$$

  and hence

$$F_{ij}^* = \frac{\partial x_i^*}{\partial X_j} = \frac{\partial x_i^*}{\partial x_k}\frac{\partial x_k}{\partial X_j} = Q_{ik}F_{kj}. \tag{4.19}$$

- Normal to the surface: Under the rotation $\boldsymbol{Q}$, the unit normal $\boldsymbol{n}$ to ∂R_t becomes

$$\boxed{\boldsymbol{n}^* = \boldsymbol{Q}\boldsymbol{n},} \tag{4.20}$$

  which can be proved by using Nanson's formula for the motion bringing particles at $\boldsymbol{x}$ to position $\boldsymbol{x}^*$. For this motion, the deformation tensor is $\partial\boldsymbol{x}^*/\partial\boldsymbol{x} = \boldsymbol{Q}$, so that $\boldsymbol{n}^* da^* = (\det\boldsymbol{Q})\boldsymbol{Q}^{-T}\boldsymbol{n}da = \boldsymbol{Q}\boldsymbol{n}da$, because $\boldsymbol{Q}$ is a rotation. Now it can be proved formally that $da^* = da$, but this is easy to see when we think of da as being the area of a small parallelogram, whose shape will not be changed by the application of a translation $\boldsymbol{c}$ and a rotation $\boldsymbol{Q}$.

- Mass density: We use the conservation of mass (3.5), which here reads $\rho^* = (\det \boldsymbol{Q})^{-1}\rho$, or

$$\boxed{\rho^* = \rho.} \tag{4.21}$$

- Cauchy-Green deformation tensors: It is easy to show, using (4.16), that

$$\boxed{\boldsymbol{B}^* = \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^T, \qquad \boldsymbol{C}^* = \boldsymbol{C}}. \tag{4.22}$$

- Velocity and acceleration: Let

$$\boldsymbol{x}^* \equiv \boldsymbol{\chi}^*(\boldsymbol{X}, t) = \boldsymbol{Q}(t)\boldsymbol{\chi}(\boldsymbol{X}, t) + \boldsymbol{c}(t). \tag{4.23}$$

Then we have for the velocity,

$$\boxed{\boldsymbol{v}^* \equiv \tfrac{\partial \boldsymbol{\chi}^*}{\partial t} = \boldsymbol{Q}\boldsymbol{v} + \dot{\boldsymbol{Q}}\boldsymbol{x} + \dot{\boldsymbol{c}},} \tag{4.24}$$

and for the acceleration,

$$\boxed{\boldsymbol{a}^* \equiv \tfrac{\partial^2 \boldsymbol{\chi}^*}{\partial t^2} = \boldsymbol{Q}\boldsymbol{a} + 2\dot{\boldsymbol{Q}}\boldsymbol{v} + \ddot{\boldsymbol{Q}}\boldsymbol{x} + \ddot{\boldsymbol{c}}.} \tag{4.25}$$

Definition: Objective fields. Let ϕ, $\boldsymbol{u}$, $\boldsymbol{T}$ be scalar, vector and (second-order) tensor fields defined on B_t, i.e. they are Eulerian in character. Let ϕ^*, $\boldsymbol{u}^*$, $\boldsymbol{T}^*$ be the corresponding fields defined on B_t^*, where B_t^* is obtained from B_t by the rigid motion $\boldsymbol{x}^* = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}$. The fields are said to be *objective* if, for all such motions,

$$\boxed{\phi^* = \phi, \qquad \boldsymbol{u}^* = \boldsymbol{Q}\boldsymbol{u}, \qquad \boldsymbol{T}^* = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T.} \tag{4.26}$$

Hence, according to our list above, $\boldsymbol{F}$ is not objective, $\boldsymbol{n}$ is objective, ρ is objective, $\boldsymbol{B}$ is objective, $\boldsymbol{C}$ is not objective (it is 'indifferent'), $\boldsymbol{v}$ and $\boldsymbol{a}$ are not objective.

Example If ϕ is an objective scalar field then $\operatorname{grad}\phi$ is an objective vector field. Proof: We note that, in components,

$$\begin{aligned}(\operatorname{grad}\phi)_i^* &= (\operatorname{grad}^*\phi^*)_i = (\operatorname{grad}^*\phi)_i \qquad (\text{since } \phi^* = \phi)\\ &= \frac{\partial \phi}{\partial x_i^*} = \frac{\partial \phi}{\partial x_k}\frac{\partial x_k}{\partial x_i^*}.\end{aligned} \tag{4.27}$$

Next, since $\boldsymbol{x}^* = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}$, it follows that $\boldsymbol{x} = \boldsymbol{Q}^T\boldsymbol{x}^* - \boldsymbol{Q}^T\boldsymbol{c}$, or, in components,

$$x_k = Q_{pk}x_p^* - Q_{pk}c_p. \tag{4.28}$$

Hence

$$\frac{\partial x_k}{\partial x_i^*} = Q_{pk}\frac{\partial x_p^*}{\partial x_i^*} = Q_{pk}\delta_{pi} = Q_{ik}, \tag{4.29}$$

which leads to

$$(\operatorname{grad}\phi)_i^* = Q_{ik}(\operatorname{grad}\phi)_k, \tag{4.30}$$

i.e. $(\operatorname{grad}\phi)^* = \boldsymbol{Q}(\operatorname{grad}\phi)$. Thus, $\operatorname{grad}\phi$ is an objective vector.

Axiom: Invariance under rigid-body motions. It is an axiom of Mechanics that the material response should be objective. In particular, the stress vector $\boldsymbol{t}_{(\boldsymbol{n})}$ must be an objective vector, and the strain energy density W must be an objective scalar function,

Hence, under the rotation $\boldsymbol{Q}$, the traction vector $\boldsymbol{t}_{(\boldsymbol{n})}$ becomes

$$\boxed{\boldsymbol{t}^*_{(\boldsymbol{n}^*)} = \boldsymbol{Q}\boldsymbol{t}_{(\boldsymbol{n})}.} \tag{4.31}$$

We call $\boldsymbol{\sigma}^*$ the stress tensor associated with the deformation gradient $\boldsymbol{F}^*$, and we now determine how $\boldsymbol{\sigma}^*$ is related to $\boldsymbol{\sigma}$. Since $\boldsymbol{t}_{(\boldsymbol{n})} = \boldsymbol{\sigma}\boldsymbol{n}$ and $\boldsymbol{t}^*_{(\boldsymbol{n}^*)} = \boldsymbol{\sigma}^*\boldsymbol{n}^*$ we obtain

$$\boldsymbol{\sigma}^*\boldsymbol{Q}\boldsymbol{n} = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{n}. \tag{4.32}$$

This holds for arbitrary $\boldsymbol{n}$ and hence $\boldsymbol{\sigma}^*\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{\sigma}$, i.e.

$$\boxed{\boldsymbol{\sigma}^* = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T.} \tag{4.33}$$

In other words, the Cauchy stress tensor $\boldsymbol{\sigma}$ is *objective.*

Now observe that since the strain energy density W is a scalar function, objectivity requires that it is unaffected by a superimposed rigid-body rotation after deformation, i.e. $W = W^*$ and $W(\boldsymbol{F}) = W^*(\boldsymbol{F}^*)$. In other words,

$$\boxed{W(\boldsymbol{F}) = W(\boldsymbol{Q}\boldsymbol{F}),} \tag{4.34}$$

for all rotations $\boldsymbol{Q}$ and for each deformation gradient $\boldsymbol{F}$. This may also be expressed by referring to W as being indifferent to observer transformations. A consequence of (4.34) is that $W(\boldsymbol{F}) = W(\boldsymbol{Q}\boldsymbol{R}\boldsymbol{U})$ where we used the polar decomposition of $\boldsymbol{F}$. In particular, it must hold when $\boldsymbol{Q} = \boldsymbol{R}^T$, which gives

$$\boxed{W = W(\boldsymbol{U}),} \tag{4.35}$$

and, by the square root theorem,

$$\boxed{W = W(\boldsymbol{C}).} \tag{4.36}$$

Conversely, if $W(\boldsymbol{F}) = W(\boldsymbol{C})$, then $W(\boldsymbol{Q}\boldsymbol{F}) = W(\boldsymbol{F}^T\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F}) = W(\boldsymbol{C}) = W(\boldsymbol{F})$. Hence, (4.34) and (4.36) are equivalent.

4.4 Material symmetry

Let $\boldsymbol{\sigma}$ be the stress in configuration B_t, and let $\boldsymbol{F}$, $\bar{\boldsymbol{F}}$ be the deformation gradients in B_t relative to the reference configurations B_r, $\bar{\mathrm{B}}_r$, respectively, as depicted in Fig. 10.

Let $\boldsymbol{P} = \mathrm{Grad}\,\bar{\boldsymbol{X}}$ be the deformation gradient of $\bar{\mathrm{B}}_r$ relative to B_r, where $\bar{\boldsymbol{X}}$ is the position vector of a point in $\bar{\mathrm{B}}_r$. Then

$$\boxed{\boldsymbol{F} = \bar{\boldsymbol{F}}\boldsymbol{P}.} \tag{4.37}$$

To prove (4.37), we use index notation. We have

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial x_i}{\partial \bar{X}_k}\frac{\partial \bar{X}_k}{\partial X_j} = \bar{F}_{ik}P_{kj}. \tag{4.38}$$

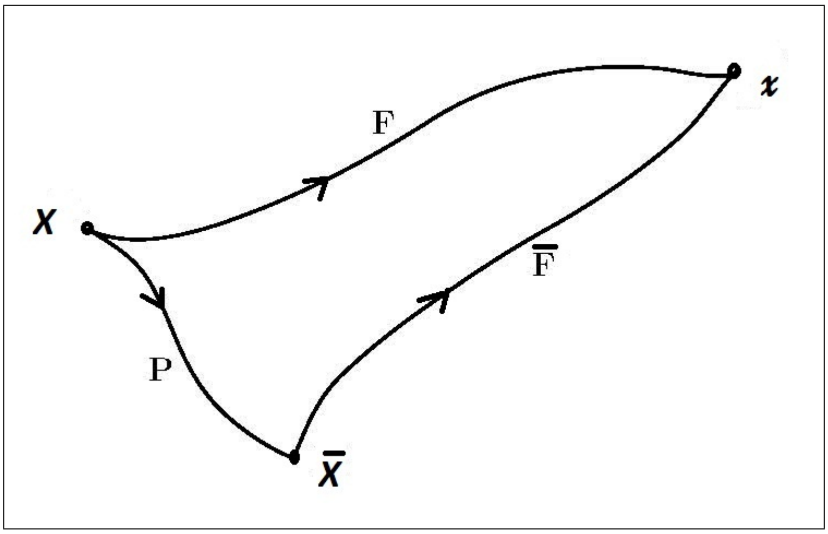

Figure 10: Paths of deformation with deformation gradients $\boldsymbol{F}$ and $\bar{\boldsymbol{F}}$ from reference configurations B_r and $\bar{\mathrm{B}}_r$, which are connected by deformation gradient $\boldsymbol{P}$.

Definition: Isotropy A material is said to be *isotropic* relative to B_r, when its mechanical behaviour is unaffected by *any rotation* $\boldsymbol{P}$ that takes place prior to a given deformation. Physically, this means that the response of a small specimen of material cut from B_r is independent of its orientation in B_r. Then we have

$$W(\boldsymbol{F}) = W(\bar{\boldsymbol{F}}) = W(\boldsymbol{F}\boldsymbol{P}^T), \tag{4.39}$$

for all rotations $\boldsymbol{P}$.

On the other hand, recall that by objectivity, $W(\boldsymbol{F}) = W(\boldsymbol{U}) = W(\boldsymbol{Q}\boldsymbol{F})$, so that

$$W(\boldsymbol{U}) = W(\boldsymbol{Q}\boldsymbol{F}\boldsymbol{P}^T), \tag{4.40}$$

for all rotations $\boldsymbol{Q}$ and $\boldsymbol{P}$. Choosing $\boldsymbol{Q} = \boldsymbol{P}\boldsymbol{R}^T$, where $\boldsymbol{R}$ is the rotation tensor in the polar decomposition of $\boldsymbol{F}$, we conclude that for isotropic hyperelastic materials,

$$W(\boldsymbol{U}) = W(\boldsymbol{P}\boldsymbol{U}\boldsymbol{P}^T), \tag{4.41}$$

for all rotations $\boldsymbol{P}$. Finally, by the square root theorem, this is equivalent to

$$\boxed{W(\boldsymbol{C}) = W(\boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^T),} \tag{4.42}$$

for all rotations $\boldsymbol{P}$.

4.5 Isotropic hyperelasticity

From the spectral decomposition theorem, we know that the right Cauchy-Green strain tensor $\boldsymbol{C}$ depends in general on its eigenvalues λ_i^2 and its eigenvectors $\boldsymbol{u}^{(i)}$ $(i = 1, 2, 3)$:

$$\boldsymbol{C} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \tag{4.43}$$

in the ($\boldsymbol{u}^{(1)}$, $\boldsymbol{u}^{(2)}$, $\boldsymbol{u}^{(3)}$) orthonormal basis. Since $W(\boldsymbol{PCP}^T) = W(\boldsymbol{C})$ for arbitrary orthogonal $\boldsymbol{P}$, we conclude that W depends on $\boldsymbol{C}$ only through its eigenvalues, which are quantities unaffected by the rotations (because $\det(\boldsymbol{PCP}^T - \lambda^2\boldsymbol{I}) = \det[\boldsymbol{P}(\boldsymbol{C} - \lambda^2\boldsymbol{I})\boldsymbol{P}^T] = \det(\boldsymbol{C} - \lambda^2\boldsymbol{I})$). Hence, we may write

$$\boxed{W(\boldsymbol{C}) \equiv W(\lambda_1, \lambda_2, \lambda_3).} \tag{4.44}$$

Now take the particular rotation about $\boldsymbol{u}^{(3)}$ such that

$$\boldsymbol{Pu}^{(1)} = \boldsymbol{u}^{(2)}, \qquad \boldsymbol{Pu}^{(2)} = -\boldsymbol{u}^{(1)}, \qquad \boldsymbol{Pu}^{(3)} = \boldsymbol{u}^{(3)}. \tag{4.45}$$

Then, in the eigenvectors orthonormal basis,

$$\boldsymbol{PCP}^T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda_2^2 & 0 & 0 \\ 0 & \lambda_1^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \tag{4.46}$$

i.e. λ_1^2 and λ_2^2 have swapped roles. It follows that $W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_2, \lambda_1, \lambda_3)$, and we may proceed similarly with other rotations about the eigenvectors. We say that W is a *symmetric* function of λ_1, λ_2, λ_3:

$$\boxed{W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_2, \lambda_1, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2) = W(\lambda_3, \lambda_2, \lambda_1).} \tag{4.47}$$

But λ_1^2, λ_2^2, λ_3^2 are the (real) roots of the cubic

$$\lambda^6 - I_1\lambda^4 + I_2\lambda^2 - I_3 = 0, \tag{4.48}$$

where

$$I_1 = \mathrm{tr}\boldsymbol{C}, \qquad I_2 = \tfrac{1}{2}\left[(\mathrm{tr}\boldsymbol{C})^2 - \mathrm{tr}\left(\boldsymbol{C}^2\right)\right], \qquad I_3 = \det\boldsymbol{C}, \tag{4.49}$$

are the principal invariants of $\boldsymbol{C}$ (and of $\boldsymbol{B}$). They are related to the λ_i through

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \qquad I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \qquad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2, \tag{4.50}$$

see (2.51). By inverting these equations, we can express the λ_i as functions of the I_i. Hence ϕ depends only on I_1, I_2, I_3:

$$\boxed{W(\boldsymbol{C}) \equiv W(I_1, I_2, I_3),} \tag{4.51}$$

or, equivalently, it depends symmetrically on λ_1, λ_2, λ_3.

5 Stress-strain relations for isotropic materials

5.1 Compressible materials

For an isotropic hyperelastic material, the strain-energy function W is expressible as a function of the principal invariants, as in (4.51): $W = W(I_1, I_2, I_3)$, and the nominal stress $\boldsymbol{S}$ is related to W through (4.12): $S_{ji} = \partial W/\partial F_{ij}$. It follows that

$$S_{ij} = \frac{\partial I_1}{\partial F_{ji}}\frac{\partial W}{\partial I_1} + \frac{\partial I_2}{\partial F_{ji}}\frac{\partial W}{\partial I_2} + \frac{\partial I_3}{\partial F_{ji}}\frac{\partial W}{\partial I_3}. \tag{5.1}$$

In order to obtain an expression for the nominal stress $\boldsymbol{S}$ we need the derivatives of the principal invariants with respect to $\boldsymbol{F}$. We find:

$$\frac{\partial I_1}{\partial \boldsymbol{F}} = 2\boldsymbol{F}^T, \qquad \frac{\partial I_2}{\partial \boldsymbol{F}} = 2I_1\boldsymbol{F}^T - 2\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{F}^T, \qquad \frac{\partial I_3}{\partial \boldsymbol{F}} = 2I_3\boldsymbol{F}^{-1}. \tag{5.2}$$

Proof: Recall that $I_1 = C_{kk} = F_{lk}F_{lk}$, so that

$$\frac{\partial I_1}{\partial F_{ji}} = 2\delta_{jl}\delta_{ik}F_{lk} = 2F_{ji}. \tag{5.3}$$

Then the proof for the second identity proceeds similarly. Finally, the proof for the third identity relies on (2.124), which reads

$$\frac{\partial}{\partial \tau}(\det \boldsymbol{F}) = (\det \boldsymbol{F})\,\mathrm{tr}\left(\boldsymbol{F}^{-1}\frac{\partial \boldsymbol{F}}{\partial \tau}\right), \tag{5.4}$$

for any scalar variable τ. Here we write

$$\begin{aligned}\frac{\partial I_3}{\partial F_{ji}} &= \frac{\partial}{\partial F_{ji}}(\det \boldsymbol{F})^2 = 2(\det \boldsymbol{F})\frac{\partial}{\partial F_{ji}}(\det \boldsymbol{F}) = 2(\det \boldsymbol{F})^2\mathrm{tr}\left(\boldsymbol{F}^{-1}\frac{\partial \boldsymbol{F}}{\partial F_{ji}}\right),\\ &= 2I_3F_{kl}^{-1}\frac{\partial F_{lk}}{\partial F_{ji}} = 2I_3F_{kl}^{-1}\delta_{jl}\delta_{ki} = 2I_3F_{ij}^{-1}.\end{aligned} \tag{5.5}$$

Going back now to (5.1), we thus have

$$\boldsymbol{S} = 2W_1\boldsymbol{F}^T + 2W_2(I_1\boldsymbol{F}^T - \boldsymbol{F}^T\boldsymbol{F}\boldsymbol{F}^T) + 2I_3W_3\boldsymbol{F}^{-1}, \tag{5.6}$$

where

$$W_1 = \frac{\partial W}{\partial I_1}, \qquad W_2 = \frac{\partial W}{\partial I_2}, \qquad W_3 = \frac{\partial W}{\partial I_3}. \tag{5.7}$$

The corresponding expression for the Cauchy stress is then found from the connection (3.36) (i.e. $\boldsymbol{\sigma} = J^{-1}\boldsymbol{F}\boldsymbol{S}$), as

$$\boldsymbol{\sigma} = 2I_3^{1/2}W_3\boldsymbol{I} + 2I_3^{-1/2}(W_1 + I_1W_2)\boldsymbol{B} - 2I_3^{-1/2}W_2\boldsymbol{B}^2, \tag{5.8}$$

where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ is the right Cauchy-Green strain tensor. Writing down the Cayley-Hamilton theorem for $\boldsymbol{B}$, we can multiply (2.57) across by $\boldsymbol{B}^{-1}$ to express $\boldsymbol{B}^2$ in terms of $\boldsymbol{B}$, $\boldsymbol{I}$, and $\boldsymbol{B}^{-1}$. Then, a stress-strain relation equivalent to (5.8) is

$$\boldsymbol{\sigma} = 2I_3^{-1/2}[I_2W_2 + I_3W_3]\boldsymbol{I} + 2I_3^{-1/2}W_1\boldsymbol{B} - 2I_3^{1/2}W_2\boldsymbol{B}^{-1}. \tag{5.9}$$

We have thus established the two equivalent forms relating the Cauchy stress to the Cauchy-Green strain:

$$\boxed{\boldsymbol{\sigma} = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{B} + \beta_2 \boldsymbol{B}^2,} \tag{5.10}$$

where

$$\beta_0 = 2I_3^{1/2} W_3, \qquad \beta_1 = 2I_3^{-1/2}(W_1 + I_1 W_2), \qquad \beta_2 = -2I_3^{-1/2} W_2, \tag{5.11}$$

and

$$\boxed{\boldsymbol{\sigma} = \chi_0 \boldsymbol{I} + \chi_1 \boldsymbol{B} + \chi_{-1} \boldsymbol{B}^{-1},} \tag{5.12}$$

where

$$\chi_0 = 2I_3^{-1/2}(I_2 W_2 + I_3 W_3), \qquad \chi_1 = 2I_3^{-1/2} W_1, \qquad \chi_{-1} = -2I_3^{1/2} W_2. \tag{5.13}$$

5.2 Stress-deformation relations in terms of stretches

Instead of using the principal invariants I_1, I_2, I_3 as independent measures of deformation, we can use (equivalently) the stretches λ_1, λ_2, λ_3. They are related by

$$\begin{aligned}
I_1 &= \mathrm{tr}\,(\boldsymbol{B}) \equiv \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\
I_2 &= \tfrac{1}{2}[I_1^2 - \mathrm{tr}\,(\boldsymbol{B}^2)] \equiv \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \\
I_3 &= \det \boldsymbol{B} \equiv \lambda_1^2\lambda_2^2\lambda_3^2 \equiv J^2,
\end{aligned} \tag{5.14}$$

and we note that these are symmetric functions of the stretches.

In the coordinate system ($\boldsymbol{v}^{(1)}$, $\boldsymbol{v}^{(2)}$, $\boldsymbol{v}^{(3)}$) of the unit eigenvectors of $\boldsymbol{B}$, we have

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \qquad \boldsymbol{B}^2 = \begin{bmatrix} \lambda_1^4 & 0 & 0 \\ 0 & \lambda_2^4 & 0 \\ 0 & 0 & \lambda_3^4 \end{bmatrix}, \tag{5.15}$$

and thus, using (5.10),

$$\sigma_{11} = \beta_0 + \beta_1 \lambda_1^2 + \beta_2 \lambda_1^4 = 2J^{-1}[I_3 W_3 + (W_1 + I_1 W_2)\lambda_1^2 - W_2 \lambda_1^4]. \tag{5.16}$$

Compare this expression with that obtained when computing $J^{-1}\lambda_1 \partial W/\partial \lambda_1$:

$$\begin{aligned}
J^{-1}\lambda_1 \frac{\partial W}{\partial \lambda_1} &= J^{-1}\lambda_1 \left[\frac{\partial I_1}{\partial \lambda_1} W_1 + \frac{\partial I_2}{\partial \lambda_1} W_2 + \frac{\partial I_3}{\partial \lambda_1} W_3 \right] \\
&= J^{-1}\lambda_1 \left[2\lambda_1 W_1 + 2\lambda_1(\lambda_2^2 + \lambda_3^2) W_2 + 2\lambda_1 \lambda_2^2 \lambda_3^2 W_3 \right] \\
&= 2J^{-1} \left[\lambda_1^2 W_1 + (\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2) W_2 + I_3 W_3 \right].
\end{aligned} \tag{5.17}$$

These expressions are indeed the same. We conclude that the principal Cauchy stresses $\sigma_i \equiv \sigma_{ii}$ are given by

$$\boxed{\sigma_i = J^{-1}\lambda_i \frac{\partial W}{\partial \lambda_i}, \qquad i \in \{1, 2, 3\},} \tag{5.18}$$

where there is no sum on the repeated indices.

5.3 Incompressible elastic materials

If the considered material is *incompressible*, then it can only accommodate isochoric deformations. The deformation gradient $\boldsymbol{F}$ must satisfy the *internal constraint*

$$\boxed{J \equiv \det \boldsymbol{F} \equiv \det \boldsymbol{U} \equiv \lambda_1\lambda_2\lambda_3 = 1,} \tag{5.19}$$

at each point of the material, at all times. This condition is equivalent to (see (2.126))

$$\boxed{\operatorname{div} \boldsymbol{v} = \operatorname{tr} \boldsymbol{D} = \operatorname{tr} \boldsymbol{L} = 0.} \tag{5.20}$$

Now recall that in order to derive the relationship between the stress and the strain in hyperelastic solids, we started from the expression (4.6) for the rate of change of W:

$$\frac{\partial W}{\partial t} = J \operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{L}) = \operatorname{tr}(\boldsymbol{S}\dot{\boldsymbol{F}}). \tag{5.21}$$

For incompressible solids, the stress power $\partial W/\partial t$ is the same for the tensor $\boldsymbol{\sigma}$ and for the tensor $\boldsymbol{\sigma} + p\boldsymbol{I}$ where p is arbitrary, because

$$J\operatorname{tr}[(\boldsymbol{\sigma} + p\boldsymbol{I})\boldsymbol{L}] = J\operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{L}) + pJ\operatorname{tr}(\boldsymbol{L}) = J\operatorname{tr}(\boldsymbol{\sigma}\boldsymbol{L}) + 0. \tag{5.22}$$

The stress-strain relation follows as $\boldsymbol{\sigma} + p\boldsymbol{I} = \beta_0\boldsymbol{I} + \beta_1\boldsymbol{B} + \beta_2\boldsymbol{B}^2$, where now

$$\beta_0 = 2\times 1\times \frac{\partial W}{\partial I_3} = 0, \qquad \beta_1 = 2\times 1\times (W_1 + I_1 W_2), \qquad \beta_2 = -2\times 1\times W_2. \tag{5.23}$$

Here $\beta_0 = 0$ because W depends on I_1 and I_2 only for incompressible materials, not on I_3 which is always fixed at $I_3 = J^2 = 1$. In conclusion, the general stress-strain relationship for incompressible isotropic hyperelastic materials is

$$\boxed{\boldsymbol{\sigma} = -p\boldsymbol{I} + 2(W_1 + I_1W_2)\boldsymbol{B} - 2W_2\boldsymbol{B}^2,} \tag{5.24}$$

where $p = p(\boldsymbol{x})$ is an arbitrary scalar (to be determined later). It can also be written as

$$\boxed{\boldsymbol{\sigma} = -\bar{p}\boldsymbol{I} + 2W_1\boldsymbol{B} - 2W_2\boldsymbol{B}^{-1},} \tag{5.25}$$

where $\bar{p} = p - 2W_2I_2$ is also arbitrary. Finally, when W is seen as a function of the stretches, the principal Cauchy stresses now read

$$\boxed{\sigma_i = -p + \lambda_i\frac{\partial W}{\partial \lambda_i}, \qquad i \in \{1,2,3\},} \tag{5.26}$$

where there is no sum on the repeated indices.

5.4 Examples of strain-energy functions

Many different strain-energy functions are available in the literature to model the behaviour of rubberlike solids and other soft materials. Here we provide a limited number of examples for *incompressible* isotropic elasticity based on use of the invariants I_1, I_2, and of the stretches λ_1, λ_2, λ_3 (subject to the constraint $\lambda_1\lambda_2\lambda_3 = 1$).

5.4.1 Use of the invariants I_1, I_2

A basic strain-energy function, known as the *neo-Hookean material*, has the form

$$\boxed{W = \frac{\mu_0}{2}(I_1 - 3),} \tag{5.27}$$

where $\mu_0 (> 0)$ is a material constant referred to as the *shear modulus* of the material in the natural configuration. This is a prototype model for rubber elasticity. The associated Cauchy stress is given by

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu_0 \boldsymbol{B}. \tag{5.28}$$

Another such model is the *Mooney-Rivlin material*, defined by

$$\boxed{W = C_1(I_1 - 3) + C_2(I_2 - 3),} \tag{5.29}$$

where $C_1 \, (\geq 0)$ and $C_2 \, (> 0)$ are constants. The corresponding Cauchy stress can be calculated from (5.25) as

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2C_1\boldsymbol{B} - 2C_2\boldsymbol{B}^{-1}. \tag{5.30}$$

The *Gent model* is often used to describe solids which stiffen rapidly as they are stretched:

$$\boxed{W = -\frac{\mu_0 J_m}{2} \ln\left(1 - \frac{I_1 - 3}{J_m}\right),} \tag{5.31}$$

where μ_0 and J_m are positive constants. For this material, the stress-strain relationship is

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \frac{\mu_0 J_m}{J_m + 3 - I_1}\boldsymbol{B}, \tag{5.32}$$

showing that the Cauchy stress will blow up as the stretches increase to the point where $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ comes near to $J_m + 3$. Hence, J_m is a "stiffening" parameter: the smaller it is, the earlier the strain-stiffening effect occurs.

Finally, the *Fung material* is a good candidate to model the behaviour of biological soft (isotropic) tissues:

$$\boxed{W = \frac{\mu_0}{2b} \mathrm{e}^{b(I_1 - 3)},} \tag{5.33}$$

where μ_0 and b are positive constants. Here the stress-strain relationship is

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + \mu_0 \mathrm{e}^{b(I_1 - 3)}\boldsymbol{B}. \tag{5.34}$$

5.4.2 Use of the stretches

An example of a strain-energy function for incompressible materials expressed as a function of the stretches is that of the *Ogden material*, given by

$$\boxed{W = \sum\nolimits_{n=1}^{N} \frac{\mu_n}{\alpha_n} \left(\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3\right),} \tag{5.35}$$

where N is a positive integer and μ_n and α_n are material constants such that

$$\mu_n \alpha_n > 0, \qquad n = 1, 2, \dots, N. \tag{5.36}$$

From (5.26) the principal Cauchy stresses are calculated as

$$\sigma_i = -p + \sum_{n=1}^{N} \mu_n \lambda_i^{\alpha_n}, \qquad i \in \{1, 2, 3\}. \tag{5.37}$$

This model is most popular for curve-fitting exercises, because its accuracy increases rapidly with N. For instance, when choosing to fit an experimental stress-stretch curve with a 3-term Ogden material, we have 6 parameters to adjust at our disposal (α_1, α_2, α_3, μ_1, μ_2, μ_3). However, problems of non-uniqueness for the best set of parameters arise also.

5.5 Application to homogeneous deformations

We recall that for a homogeneous deformation the deformation gradient $\boldsymbol{F}$ is constant, i.e. independent of position $\boldsymbol{X}$. Then lines that were parallel in the reference configuration remain parallel in the deformed configuration.

A *pure homogeneous strain* is a deformation of the form

$$x_1 = \lambda_1 X_1, \qquad x_2 = \lambda_2 X_2, \qquad x_3 = \lambda_3 X_3, \tag{5.38}$$

where the principal stretches $\lambda_1, \lambda_2, \lambda_3$ are constants and (X_1, X_2, X_3) and (x_1, x_2, x_3) are Cartesian coordinates. For this deformation $\boldsymbol{F} = \boldsymbol{U} = \boldsymbol{V}$, $\boldsymbol{R} = \boldsymbol{I}$ and the principal axes of the deformation coincide with the Cartesian coordinate directions, i.e. they do not change their orientation as the values of the stretches change.

A *simple shear* is a deformation of the form

$$x_1 = X_1 + K X_2, \qquad x_2 = X_2, \qquad x_3 = X_3, \tag{5.39}$$

where the constant K is the amount of shear. Here the orientations of the principal axes of the deformation change as the amount of shear changes.

5.5.1 Simple tension

The *simple tension* test is the most widely available method of testing for soft materials. A rectangular sample is clamped at both ends, and a tensile machine imposes a linear displacement of the clamps, while measuring the force and the stretch during the extension. By dividing the force with the original cross-section area of the sample, we gain access to the nominal stress S_{11}. By multiplying this quantity with the stretch, we obtain the principal Cauchy stress σ_1. This test has the advantage that relatively large values of the stretches can be achieved.

Simple tension leads to simple extension, a special case of the pure homogeneous deformation (5.38). Call λ the stretch in the direction of tension: $\lambda = \lambda_1$, say. By symmetry, the lateral stretches are equal: $\lambda_2 = \lambda_3$. By the incompressibility condition, $\lambda_1\lambda_2\lambda_3 = \lambda\lambda_2^2 = 1$. It follows that the principal stretches of simple extension are:

$$\lambda_1 = \lambda, \qquad \lambda_2 = \lambda^{-1/2}, \qquad \lambda_3 = \lambda^{-1/2}. \tag{5.40}$$

Also, in simple tension we have $\sigma_1 \neq 0$ and $\sigma_2 = \sigma_3 = 0$, so that by (5.25),

$$\begin{aligned} \sigma_1 &= -p + 2W_1\lambda^2 - 2W_2\lambda^{-2}, \\ \sigma_2 &= \sigma_3 = 0 = -p + 2W_1\lambda^{-1} - 2W_2\lambda. \end{aligned} \tag{5.41}$$

Subtracting one equation from the other we find that in *simple tension*,

$$\boxed{\sigma_1 = 2(\lambda^2 - \lambda^{-1})W_1 - 2(\lambda^{-2} - \lambda)W_2.} \tag{5.42}$$

Representative simple tension data are shown in Fig. 11 for a vulcanized natural rubber. The data are compared with the theory based on the neo-Hookean material of the form (5.27) (dashed curves) and a three-term Ogden material of the form (5.35) (continuous curve). Clearly, the neo-Hookean model, with its sole parameter to be adjusted μ_0, gives a poor fit and cannot capture the full mechanical behaviour of the rubber sample. The 3-term Ogden model has 6 parameters and can thus produce a much better fit. However there are several sets of 6 parameters giving an equally good fit, each predicting different behaviour for other deformations, which is problematic.

In Figures 12 we plot the theoretical graphs of the tensile stress in simple tension for the other models. We see that they all provide an improvement on the neo-Hookean model (dashed curves), because they have two parameters. The Mooney-Rivlin model and the Gent models both give an S-shaped curve, suitable for the modelling of rubbers and silicones, whilst the Fung model gives a J-shaped curve, typical of biological soft tissues.

5.5.2 Simple shear

Experimental tests such as biaxial deformation and simple tension are such that the principal axes of strain do not change as the magnitude of the strain is varied. We

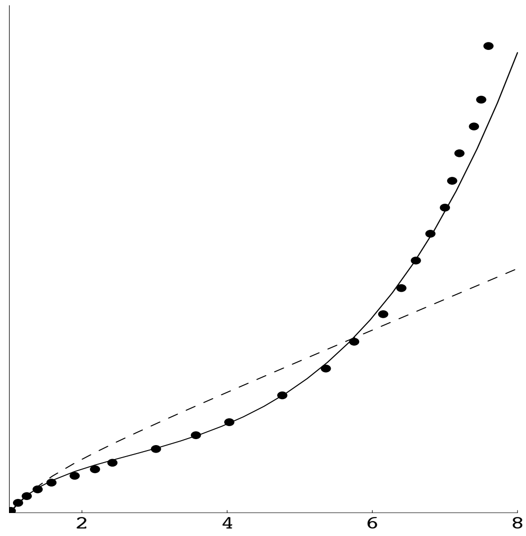

Figure 11: Simple tension data with the nominal stress (dimensionless) plotted on the vertical axis against the stretch λ for a vulcanized natural rubber (circles) compared with the predictions of the neo-Hookean material (dashed curve) and of the Ogden material with $N = 3$ (continuous curve).

now consider the predictions of the theory for a deformation for which the orientation of the principal axes of strain *does* change: the simple shear deformation.

From the simple shear deformation (5.39), we find

$$\boldsymbol{B} = \begin{bmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & -K & 0 \\ -K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.43}$$

so that the invariants I_1, I_2, I_3 are given by

$$I_1 = I_2 = 3 + K^2, \qquad I_3 = 1 \tag{5.44}$$

emphasizing that simple shear is an *isochoric* deformation. Simple shear is an important deformation since it arises locally in many problems of practical and theoretical

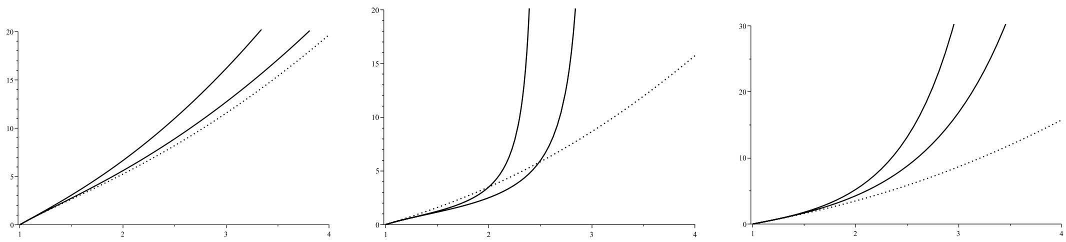

Figure 12: Tensile stress for the Mooney-Rivlin material (left), the Gent material (middle) and the Fung material (right), for different values of the parameters. The neo-Hookean curve is the dashed curve.

interest.

Using (5.25), we find the following components for $\boldsymbol{\sigma}$,

$$\begin{aligned} &\sigma_{11} = -p + 2W_1(1+K^2) - 2W_2, && \sigma_{12} = 2(W_1+W_2)K, \\ &\sigma_{22} = -p + 2W_1 - 2W_2(1+K^2), && \sigma_{13} = \sigma_{23} = 0, \\ &\sigma_{33} = -p + 2(W_1 - W_2). && \end{aligned} \tag{5.45}$$

First of all, we see notice that W depends on K^2 only, according to (5.44), so that W_1 and W_2 are functions of K^2 also. It follows that the *shear stress* σ_{12} is related to the amount of shear K in an *odd* manner: $\sigma_{12}(-K) = -\sigma_{12}(K)$ or, in other words, the shear stress required to produce the amount of shear K is the opposite of the shear stress required to produce the amount of shear $-K$. We emphasise this important result by defining the *shear modulus of nonlinear elasticity*,

$$\boxed{\sigma_{12} = \mu(K^2)K, \qquad \text{where} \quad \mu(K^2) \equiv 2(W_1+W_2)} \tag{5.46}$$

is a function of K^2 alone.

Examples: For the neo-Hookean and the Mooney-Rivlin materials, $\mu(K^2)$ is a *constant* ($\mu(K^2) = \mu_0$ and $\mu(K^2) = 2(C_1 + C_2)$, respectively). For the Gent and the Fung materials, it is an increasing function of the amount of shear ($\mu(K^2) = \mu_0 J_m/(J_m - K^2)$ and $\mu(K^2) = \mu_0 \exp(bK^2)$, respectively). In Figure 13, shear stress data is reported for brain tissue. Some authors have tried to fit this type of curve to a law of the form $\sigma_{12} = aK^b$, where a and b are fitting parameters. Clearly, this is unphysical, because it does not respect the odd relationship between shear stress and amount of shear (unless b is an odd integer). Instead, we can look closely at the curve and discard the part beyond $K > 1$, because it corresponds to an angle of shear greater than $\tan^{-1}(1) = 45°$. Beyond that angle it is safe to assume that the shear is not simple and that the deformation is no longer homogeneous (originally parallel lines do not remain parallel). Then we notice that in the range $0 < K < 1$ the stress-shear relationship is almost linear, and thus conclude that the Mooney-Rivlin material is a good candidate to model the behaviour of this sample.

Now we go back to the stress components (5.45) and notice the connection

$$\boxed{\sigma_{11} - \sigma_{22} = K\sigma_{12}.} \tag{5.47}$$

This is important to note since it is an example of a *universal relation*, i.e. a connection between the stress components that is independent of the form of constitutive law (in this case, the class of incompressible isotropic elastic solids). It follows from the universal relation that the normal stresses σ_{11} and σ_{22} can never be equal: this is called the *Poynting effect*.

Finally, we consider the case of plane stress: $\sigma_{33} = 0$. According to (5.45), this leads to $p = 2(W_1 - W_2)$ and

$$\sigma_{11} = 2W_1K^2, \qquad \sigma_{22} = -2W_2K^2. \tag{5.48}$$

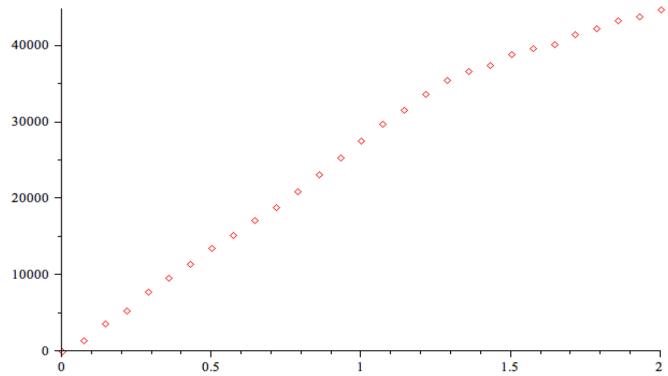

Figure 13: Cauchy shear stress σ_{12} against the amount of shear K for the large simple shear of pig brain matter.

Hence, the normal stresses σ_{11} and σ_{22} cannot be zero simultaneously (Otherwise W would not depend on I_1 nor I_2). In general *normal stresses* are required in addition to shear stresses in order to maintain the shape of the sheared material. The necessity for normal forces is an example of the *Kelvin effect*.

Going back to our example where we tried to model the behaviour of porcine brain tissue. We saw that σ_{12} is related to K in a linear manner. Hence, W is either of the neo-Hookean or of the Mooney-Rivlin type, for which we find that

$$\sigma_{22} = 0, \qquad \sigma_{22} = -2C_2K^2, \tag{5.49}$$

respectively, according to (5.27), (5.29) and (5.46). It follows that no forces normal to the platens appear in the neo-Hookean case, while they do appear in the Mooney-Rivlin case. In the experiments, we now drill a hole in the upper platen and observe that the brain matter bulges through the hole. It follows that $\sigma_{22} \neq 0$, and thus that brain matter must be described by the Mooney-Rivlin model.

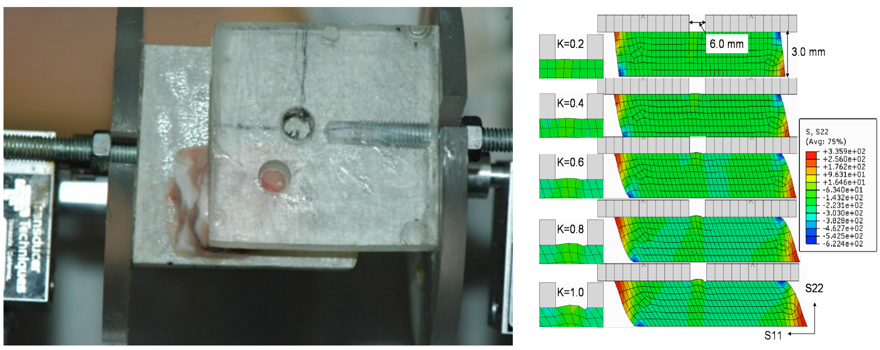

Figure 14: Normal stress arising in the simple shear of porcine brain matter.